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On the Stability of Cournot Equilibrium when the Number of Competitors Increases

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Abstract

This article reconsiders whether the Cournot equilibrium really becomes a perfect competition equilibrium when the number of competitors goes to infinity. It has been questioned whether the equilibrium remains stable with an increasing number of firms. Contraindications were given for linear and for isoelastic demand functions. However, marginal costs were then taken as constant, which means adding more potentially infinite-sized firms. As we want to compare cases with few large firms to cases with many small firms, the model is tuned so as to incorporate capacity limits, decreasing with an increasing number of firms. Then destabilization is avoided.

1 Introduction

An intriguing question in microeconomics has been whether an increase in the number of competitors in a market always defines a path from monopoly, over duopoly, oligopoly, and polypoly using Frisch's term (see Frisch 1933), to perfect competition. There are two different issues involved: (i) the Cournot equilibrium (see Cournot 1838) must have the competitive equilibrium as its limit, and (ii) the increasing number of competitors must not destabilize that equilibrium state. As a rule the first question is responded to in the affirmative, whereas there have been raised serious doubts about the second; examples are Palander (1939), Theocharis (1959), Agiza (1998), and Ahmed and Agiza (1998). Theocharis pointed out that an oligopoly system with n competitors, producing under constant marginal cost, and facing a linear demand function, would be only neutrally stable for 3 competitors and unstable for 4 and more competitors. This paradox has never been resolved, and is normally associated with the name of Theocharis.

The argument is very simple. With a linear demand function, the total revenue function for each competitor becomes quadratic in the supply of the firm itself and linear in the supplies of the competitors. Differentiating the profit function to obtain the marginal profit condition and solving for the firm's own supply as a function of the supplies of the competitors (for the reaction function) results in a linear function with the constant slope $-\frac{1}{2}$. Hence, the n

by n Jacobian matrix, whose eigenvalues determine the stability of the Cournot point, has the constant $-\frac{1}{2}$ in all off-diagonal elements, and 0 in the diagonal elements.

Accordingly, the characteristic equation factorizes into

$$\left(\lambda - \frac{1}{2}\right)^{n-1} \left(\lambda + (n-1)\frac{1}{2}\right) = 0,$$

so, the system has $n-1$ eigenvalues $\lambda_{1,\ldots(n-1)} = \frac{1}{2}$, and one eigenvalue $\lambda_n = -(n-1)\frac{1}{2}$. This last eigenvalue causes the trouble: For $n = 3$ we have $\lambda_3 = -1$, for $n = 4$ we have $\lambda_4 = -\frac{3}{2} < -1$, and so forth.

Theocharis's argument was in fact proposed 20 years earlier by Palander (p. 237), who wrote: "*as a condition for an equilibrium with a certain number of competitors to be stable to exogenous disturbances, one can stipulate that the derivative of the reaction function f' must be such that the condition $|(n-1)f'| < 1$ holds. If this criterion is applied to, for instance, the case with a linear demand function and constant marginal costs, the equilibria become unstable as soon as the number of competitors exceeds three. Not even in the case of three competitors will equilibrium be restored, rather there remains an endless oscillation*".

It is difficult to say to what extent Palander's argument was widely known, but he had already presented a substantial part of it at a Cowles Commission conference in 1936.

Linear reaction functions (arising with linear demand functions) are easy to use in the argument because their slopes are constant, so one does not need to be concerned about the argument values in the Cournot equilibrium point. Linear functions are, of course, globally a problem, because, to get things proper, one has to state them as piecewise linear in order to avoid negative supplies and prices, but neither Palander nor Theocharis were concerned with anything but local stability.

However, the same properties were shown to hold by Agiza (1998), and Ahmed and Agiza (1998) for a nonlinear (isoelastic) demand function and, again, constant marginal costs, suggested by the present author, Puu (1991). The model was originally suggested as a duopoly, Puu (1991), and then as a triopoly Puu (1996), and the focus was on the global complex dynamics and the bifurcations it gave rise to. Now the derivatives of the reaction functions are no longer constant but vary with the coordinates, and hence with the location of the Cournot point. However, it can easily be shown that if we assume the firms to be identical, then the problem of local stability becomes almost as simple as with linear demand functions. With n competitors, the derivative of the reaction functions in the Cournot point, the off-diagonal element in the Jacobian matrix, becomes $-\frac{(n-2)}{(n-1)}\frac{1}{2}$, and we get eigenvalues $\lambda_{1,\ldots(n-1)} = \frac{n-2}{n-1}\frac{1}{2}$ and $\lambda_n = -(n-2)\frac{1}{2}$, so it is now the case $n = 4$ that is neutrally stable, whereas $n = 5$ and higher become unstable.

The assumption of identical firms also eliminates a problem that does not arise in the isoelastic case, but in many others, such as, the multiple intersection points of the reaction functions, studied by Palander and Wald (see Palander 1939 and Wald 1936; see also Puu and Sushko 2002, pp. 111-146).

The result that the Cournot equilibrium becomes destabilized by an increasing number of competitors is a bit uncomfortable as it is against economic intuition. The paradox, as we will see, is due to the fact that the cases compared with few and many competitors have not been posed properly. In both the Theocharis and the Agiza cases, the firms are assumed to produce with constant marginal costs. But here lies the problem. Any firm producing with constant marginal cost is potentially infinitely large as there are no diminishing returns at all. We do not just want to add more large firms to the market; we want to compare cases with few large firms to cases with many small firms. This is impossible to model without introducing capacity limits, a fact on which already Edgeworth insisted (see Edgeworth 1897).

The present author tried his hands on introducing production cost functions with capacity limits twice before, in Puu and Norin (2003) and Puu and Ruiz (2006), but still focusing on two and three competitors and the global dynamics. Further, the cost function used made marginal cost dependent on the capacity limit and would not easily lend itself to the discussion of what happens when just the number of competitors increases. In the present setup another cost function, which also has a foundation in basic microeconomics, is therefore proposed.

2 The Model

2.1 Price and Revenue

As in Puu (1991), Puu (1996), Puu and Norin (2003), and Puu and Ruiz (2006), an isoelastic demand function is assumed.[1] Its inverse reads:

$$p := \frac{R}{\sum_{i=1}^{i=n} q_i}, \tag{1}$$

where p denotes market price, the q_i denote the supplies of the n competitors, and R denotes the sum of the (constant) budget shares that all the consumers spend on this particular commodity. It can be normalized to unity without loss of generality, but we prefer to keep it because it clarifies the interpretation of some conditions later.[2]

[1] An isoelastic demand function was also used to study the stability conditions for a general Cournot oligopoly by Chiarella and Szidarovszky (2002). They assumed a general type of cost functions with increasing marginal cost, but no capacity limits. Their very neat analysis, however, dealt with a model cast as a differential equation system, not as an iterated map, which is in focus of the present analysis. Further, they stated the general stability conditions, but did not deal with the effect of an increase in the number of competitors.

[2] Recall that an ioselastic demand function always results when the consumers maximize utility functions of the Cobb-Douglas type, for the j:th consumer: $\left(D_1^j\right)^{\alpha_1^j} \cdot \left(D_2^j\right)^{\alpha_2^j} \cdot ...$, subject to the budget constraint $y^j = p_1 D_1^j + p_2 D_2^j + ...$, where the p_k denote the prices of the commodities and D_k^j denote the quantities demanded. The well-known outcome of this

2.1.1 Some Definitions

In order to make the formulas succinct we also define *total market supply*,

$$Q := \sum_{i=1}^{i=n} q_i, \tag{2}$$

and *residual market supply*,

$$Q_i := Q - q_i. \tag{3}$$

Using (2) and (3) in (1), we obtain price as

$$p = \frac{R}{Q_i + q_i},$$

and total revenue for the i:th competitor as

$$R_i = \frac{Rq_i}{Q_i + q_i}; \tag{4}$$

whence the marginal revenue function

$$\frac{dR_i}{dq_i} = \frac{RQ_i}{(Q_i + q_i)^2}. \tag{5}$$

Note that the maximum total revenue obtainable for any firm according to (4) is R (obtained for $Q_i = 0$). This also holds for the aggregate of all firms.

2.2 Cost Functions

As for cost, assume

$$C_i := c_i \frac{u_i^2}{u_i - q_i}, \tag{6}$$

where u_i denotes the capacity limit for the i:th competitor, and c_i denotes the initial marginal production cost. If we calculate $\lim_{q_i \to 0} C_i = c_i u_i$, we realize that the product represents fixed cost. Hence, given any scaling factor c_i for marginal production cost, the fixed costs increase with production capacity, which is satisfactory for intuition. On the other hand there is no limit for C_i, as $u_i \to \infty$, because the fixed costs tend to become infinite, even though marginal costs approach a constant value. Note that (6) only makes sense for $0 < q_i \leq u_i$.

constrained maximization is $p_k D_k^j = \alpha_k^j y^j$ (i.e., that each consumer spends the fixed share α_k^j of income y^j on the k:th commodity). Obviously demand for each consumer then is reciprocal to price (i.e., $D_k^j = \frac{\alpha_k^j y^j}{p_k}$). As we are concerned with only one market, we can drop the commodity index k and sum over all the different consumers, obtaining aggregate demand as $D = \sum_j D^j = \frac{\sum_j \alpha^j y^j}{p} = \frac{R}{p}$. Obviously, $R = \sum_j \alpha^j y^j$ represents the sum over all consumers of their expenditures on this commodity. In (1) we substituted total supply $\sum_{i=1}^{i=n} q_i$ for total demand, but this is an equality that has to hold in equilibrium.

For higher output we just get another branch of the hyperbola (6), which lacks any economic meaning. In Fig. 1 we display some examples of shapes for the cost function family (6), assuming $u_i = 1, 2, 3, 4, 5$ all distinguished in different shade. As we see, in each case the u_i value establishes a vertical asymptote, at the approach of which total cost goes to infinity. We also see that the fixed costs, $c_i q_i$ (i.e., the vertical axis intercepts), are different for the different cost functions, higher the higher the capacity assumed. On the other hand the cost functions increase less sharply with output the higher the capacity limit. As a consequence, each of the five cost functions has its own range of outputs where it is most favourable, as indicated by the vertical line segments in Fig. 1.

There is one more fact to the cost function (6) worth mentioning: It contains a long-run cost function. Suppose production capacity u_i to be subject to choice in the long run. Differentiating (6) with respect to u_i, we readily obtain $\frac{\partial C_i}{\partial u_i} = c_i \frac{u_i(u_i - 2q_i)}{(u_i - q_i)^2}$, so as long as the firm produces less than half of its capacity limit, it would be advantageous to choose a lower capacity, if it produces more than half, it would be advantageous to choose a higher capacity. The best choice of capacity is at twice the actual level of production $u_i = 2q_i$, obtained putting the derivative equal to zero. Using this in (6), we obtain the linear long-run cost function $C_i^* = 4c_i q_i$. It is shown in Fig. 1 as tangent (i.e., an envelope), to all the short-run cost functions. In the present discussion we take capacities as fixed, but once we want to discuss the entry of a new firm on the market, or restructuring an old one, the cost function family assumed is very convenient.[3]

We also easily obtain marginal production cost:

$$\frac{dC_i}{dq_i} = c_i \frac{u_i^2}{(u_i - q_i)^2} \tag{7}$$

Further, $\lim_{q_i \to 0} \frac{dC_i}{dq_i} = c_i$ for any u_i, and $\lim_{u_i \to \infty} \frac{dC_i}{dq_i} = c_i$ for any q_i, so, any marginal cost function starts off with c_i for zero production, and when the capacity limit

[3] Is is worth mentioning that this type of cost function, which can look *ad hoc*, can be derived from a CES production function: $q = \left(k^{-\rho} + l^{-\rho}\right)^{-1/\rho}$. It is not restrictive to write this in symmetric form. For econometric use (see Arrow et al. 1961), it may be good to have as many parameters to vary as possible, but for analysis it is good to reduce their number by a simple linear change of scale.

In what follows we assume $\rho > 0$. Normally, $\rho < 0$ is assumed (i.e., with the isoquants meeting the axes), though Arrow et al. briefly also discuss the opposite case. They claim that the isoquants then go asymptotically to the axes, which is wrong; the asymptotes are located at distance from the axes in the positive quadrant.

This fact is exactly what one can use to obtain a short run cost function with capacity limit. In addition we only need to revive the "putty/clay" ideas from growth theory of the 70s (see Johansen 1972). However, we presently do not assume that both capital stock and manpower are fixed, just the first through some act of investment.

Denoting capital rent and wage rate r, w as usual, we get the cost function in the long run $C^* = \left(r^{\rho/(\rho+1)} + w^{\rho/(\rho+1)}\right)^{(\rho+1)/\rho} q$, and in the short run $C = \overline{k}\left(r + wq/\left(\overline{k}^{\rho} - q^{\rho}\right)^{1/\rho}\right)$ where $\overline{k}$ denotes the fixed capital stock. It is obvious that with $\rho > 0$, short run marginal cost is topologically equivalent to the assumed cost function (i.e., obtained for $\rho = 1$, where $c_i = w$ and $u_i = \overline{k}$).

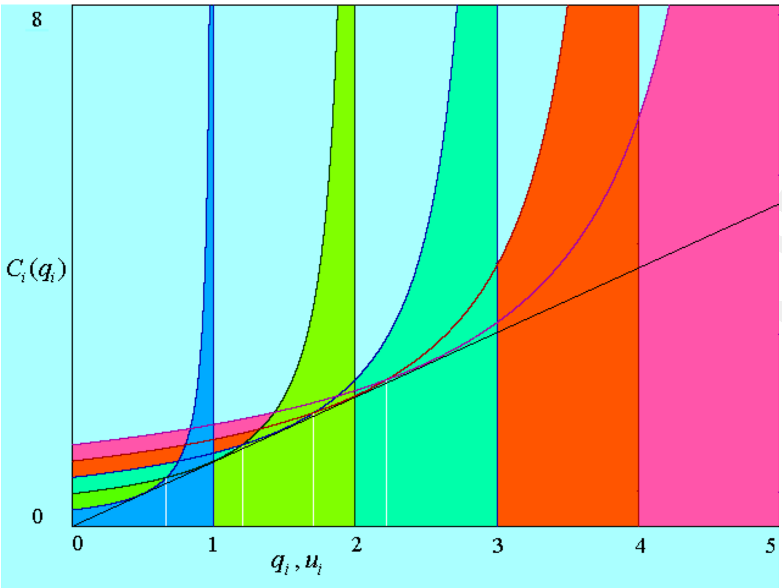

Figure 1: Total cost functions for different capacity limits.

goes to infinity, the marginal cost remains at this constant value c_i for any production level. Like the total cost functions the marginal cost functions go to infinity at the asymptote, though they all start at the same vertical axis intercept c_i. In Fig. 2 just one case of a marginal cost function is shown.

2.3 Profit Maximum and the Reaction Function

As profits are the difference between total revenue R_i and total cost C_i (i.e., $\Pi_i = R_i - C_i$) the necessary condition for profit maximum is that $\frac{d}{dq_i}\Pi_i = 0$ (i.e., that marginal revenue $\frac{dR_i}{dq_i}$ be equal to marginal cost $\frac{dC_i}{dq_i}$). From (5) and (7) we thus get

$$\frac{RQ_i}{(Q_i + q_i)^2} = c_i \frac{u_i^2}{(u_i - q_i)^2}, \tag{8}$$

or given that all variables and constants are nonnegative,

$$\frac{\sqrt{RQ_i}}{Q_i + q_i} = \sqrt{c_i} \frac{u_i}{u_i - q_i}. \tag{9}$$

Note that we always deal with a profit maximum, as $\frac{d^2}{dq_i^2}\Pi_i = \frac{d^2R_i}{dq_i^2} - \frac{d^2C_i}{dq_i^2}$, where $\frac{d^2R_i}{dq_i^2} = -2\frac{RQ_i}{(Q_i+q_i)^3} < 0$ and $\frac{d^2C_i}{dq_i^2} = 2\frac{cu^2}{(u_i-q_i)^3} > 0$, whence $\frac{d^2}{dq_i^2}\Pi_i < 0$. The profit function is convex for the variable ranges we consider, so we do not even need to substitute for the appropriate q_i from the condition (9). In

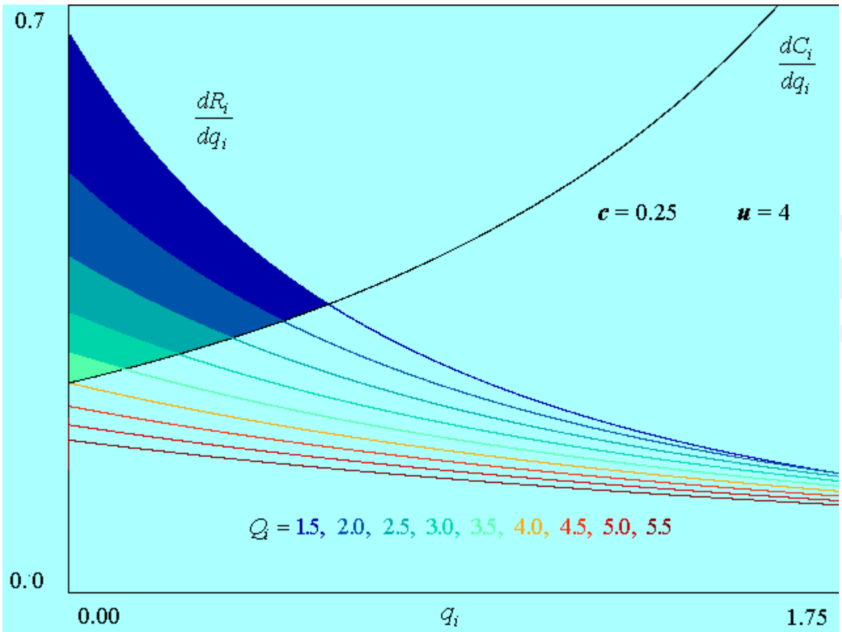

Figure 2: Marginal cost and marginal revenues for different residual supplies, with profit areas.

Fig. 2 we illustrate one marginal cost function and a family of marginal revenue functions for different residual supplies, $Q_i = 1.5$, to 5.5, with step 0.5, the curve sinking as the residual supply increases. As we see, positive profits result when $Q_i < 4.0$, and, as indicated by the shaded areas representing successive profit increments, they increase the smaller the residual supply by the competitors is.

From (9) we can easily solve for the optimal value of q_i, the supply of the i:th competitor, as a function of Q_i, the residual supply of all the other competitors. This is Cournot's reaction function:

$$q_i' = f_i(Q_i) := u_i \frac{\sqrt{RQ_i} - \sqrt{c_i}Q_i}{\sqrt{RQ_i} + \sqrt{c_i}u_i}. \tag{10}$$

Note that we anticipate considerations of dynamics by putting a dash on the optimal value q_i', as is usual in dynamical systems when we do not want to burden the notation with explicit time indices. As we know from (2) and (3), $Q_i = \sum_{j=1}^{j=n} q_j - q_i$, so from all the q_i we can calculate all the Q_i at any time, and then use (10) to calculate the q_i' and so also Q_i'. In this way the n-dimensional system is advanced step by step to obtain its orbit.

Note that from (10) we always have $q_i' < u_i$ because the quotient $\frac{\sqrt{RQ_i}-\sqrt{c_i}Q_i}{\sqrt{RQ_i}+\sqrt{c_i}u_i}$ is always less than unity. Further, of course, negative outputs make no sense, so we should make sure that (10) does not return a negative q_i'. To this end $Q_i \leq \frac{R}{c_i}$

must hold, so we should write $q_i' = \max(f_i(Q_i), 0)$ in stead of (10). However, we still have another constraint for the applicability of (10) to consider: The profit must be positive; otherwise in the long run the competitor will drop out and produce nothing, and, as we will soon see, this constraint is always at least as strong as the condition for nonnegativity of output, $Q_i \leq \frac{R}{c_i}$. This fact also implies that the curved reaction function (10) is not just replaced by the axis where it would otherwise cut the axis; it as a rule drops down to the axis. Our true reaction function is hence not just non-smooth, it also has a discontinuity.

Profits are, as we know, $\Pi_i = R_i - C_i$, so, from (4) and (6),

$$\Pi_i = \frac{Rq_i}{Q_i + q_i} - c_i \frac{u_i^2}{u_i - q_i}. \tag{11}$$

Substituting for optimal reaction from (10), we next get maximum profit

$$\Pi_i^* = c_i \frac{R - c_i u_i - 2\sqrt{c_i}\sqrt{RQ_i}}{u_i + Q_i},$$

which is nonnegative if $R - c_i u_i - 2\sqrt{c_i}\sqrt{RQ_i} \geq 0$, or, given, $c_i u_i < R$, if

$$Q_i \leq \frac{(R - c_i u_i)^2}{4Rc_i}. \tag{12}$$

As we have seen, fixed cost is $c_i u_i$, whereas R is the maximum obtainable revenue. It is hence clear that if the fixed costs alone are larger than any obtainable revenue, there can never be any positive profits.

Hence, we reformulate the map (10) as

$$q_i' = F_i(Q_i) := \begin{cases} u_i \frac{\sqrt{RQ_i} - \sqrt{c_i} Q_i}{\sqrt{RQ_i} + \sqrt{c_i} u_i} & Q_i \leq \frac{(R - c_i u_i)^2}{4Rc_i} \\ 0 & Q_i > \frac{(R - c_i u_i)^2}{4Rc_i} \end{cases}. \tag{13}$$

It remains to show that (12) is more restrictive than the nonnegativity condition $Q_i \leq \frac{R}{c_i}$ stated above. The assertion is true when

$$\frac{R}{c_i} - \frac{(R - c_i u_i)^2}{4Rc_i} = \frac{(R + c_i u_i)(3R - c_i u_i)}{4Rc_i} > 0.$$

As $c_i u_i < R < 3R$, this is certainly true, and we can content ourselves with just stating positivity of profits as a branch condition in (13).

2.4 Properties of the Reaction Functions

The reaction functions $q_i' = f_i(Q_i)$ as stated in (10) contain one positive square root term of Q_i and one negative linear term of Q_i in the numerator. The denominator does not count; it is always positive and just scales everything down when Q_i increases, so it has no importance at all for the qualitative picture of the reaction function. Due to the combination of positive lower order

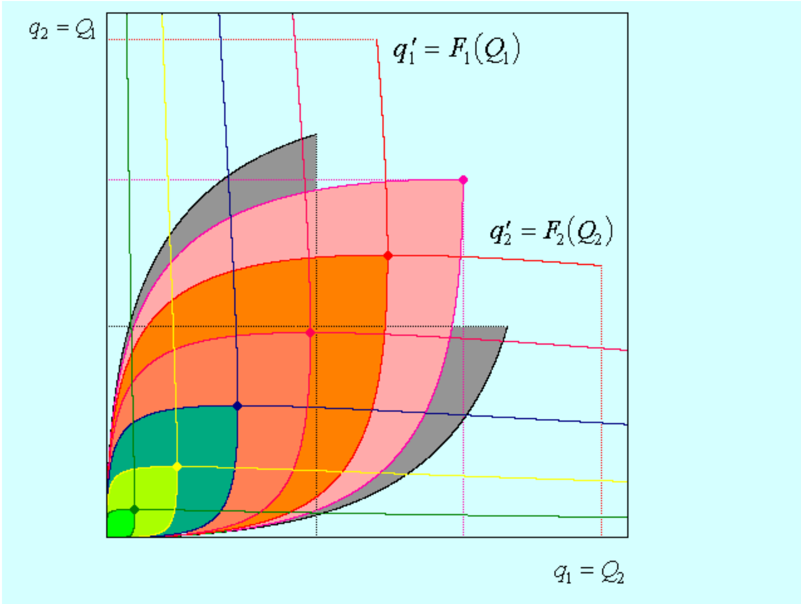

Figure 3: Reaction functions for two identical competitors and varying fixed costs.

term with negative higher order term, the result is a unimodal shape, starting in the origin, having a unique maximum, and then dropping down to the axis. Before that, at the point $Q_i = \frac{(R-c_i u_i)^2}{4c_i}$, there is a discontinuity, where the function drops down to the axis (because profits cease to be positive) as stated in (13).

In addition to this we want to know the slope (i.e., the derivative of (10)). A straightforward calculation gives us

$$\frac{df_i}{dQ_i} = \frac{\sqrt{c_i}u_i}{2} \frac{R(u_i - Q_i) - 2\sqrt{c_i}u_i\sqrt{Q_i}}{\left(\sqrt{RQ_i} + \sqrt{c_i}u_i\right)^2 \sqrt{RQ_i}}. \tag{14}$$

The numerator is zero at the maximum point, and the discontinuity can occur either before or after this maximum point. A straightforward calculation shows that this depends of whether $c_i u_i < \frac{R}{3}$ or $c_i u_i > \frac{R}{3}$. We may recall that $c_i u_i$ represents the fixed cost.

It is useful to note that

$$\lim_{Q_i \to 0} \frac{df_i}{dQ_i} = \infty.$$

Hence, all reaction functions start at the origin with an infinite slope. As all reaction functions intersect in the origin, it is good to know that this equilibrium point is totally unstable. Fig. 3 illustrates a family of reaction functions for the case of two identical competitors, assuming different fixed costs $c_i u_i$ for the firms. As we see, for lower fixed costs the reaction functions intersect in the $q_1 = Q_2$, $q_2 = Q_1$-plane. For higher fixed costs, no intersection exists, and hence no Cournot equilibrium.

3 Cournot Equilibrium

3.1 The General Case

For the equilibrium point in the general case we would just identify $q_i = q_i' = F_i(Q_i)$ according to (13), or rather, $q_i = q_i' = f_i(Q_i)$ according to (10) if we are just interested in Cournot equilibria where all firms stay active. In addition, we use $Q := \sum_{i=1}^{i=n} q_i$ from (2) and $Q_i := Q - q_i$ from (3). The resulting algebraic system of $2n+1$ equations in the variables q_i, Q_i, and Q, is well defined, but it is too awkward to provide any closed form solutions.

3.2 The Case of Identical Firms

Therefore we simplify by assuming the n firms to be identical. The only distinguishing properties of the firms are the initial marginal costs c_i and the capacity limits u_i, so let us put all marginal costs equal:

$$c_i = c \tag{15}$$

and further

$$u_i = \frac{1}{n}U \tag{16}$$

where U denotes the total capacity of the whole industry. In this way the individual capacities as well become equal, but they decrease with the number of competitors. We can see what happens when n increases, so that a given total capacity is shared among more numerous firms. Fig. 3 illustrates the case for two competitors (i.e., a duopoly).

Obviously, the identical firms produce equal quantities in Cournot equilibrium, so

$$q_i = \frac{1}{n}Q \tag{17}$$

and

$$Q_i = \frac{n-1}{n}Q \tag{18}$$

Using (15)-(18) to substitute for c_i, u_i, q_i, and Q_i in the optimum condition (8), and simplifying, we obtain

$$\frac{n-1}{n}\frac{R}{Q} = \frac{cU^2}{(U-Q)^2} \tag{19}$$

as an equation in the aggregate output Q alone. Once we have solved for Q, we can get all the individual variables q_i and Q_i from (17)-(18). The explicit

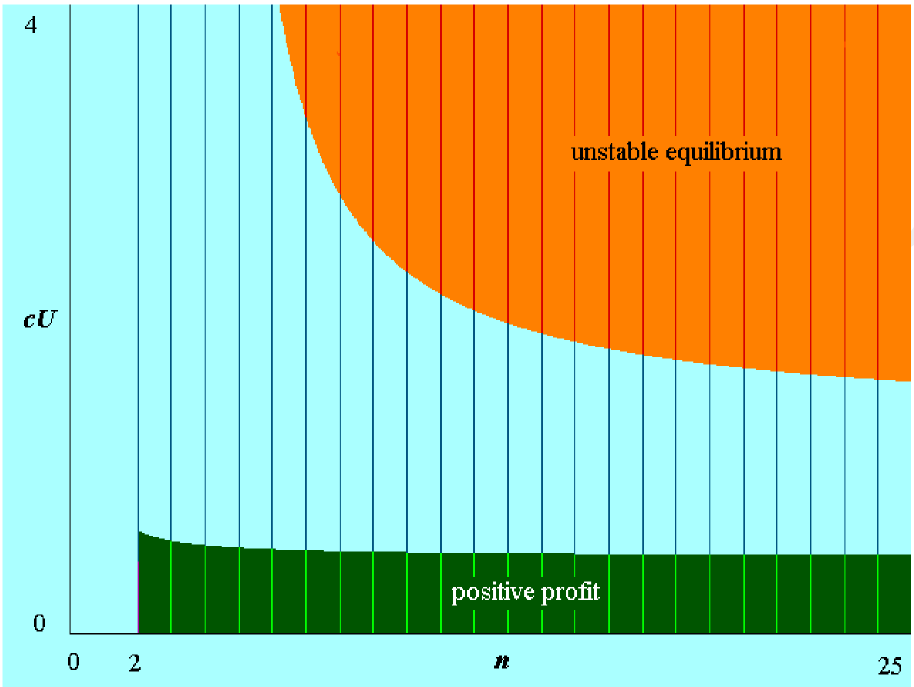

Figure 4: Areas of destabilization of the Cournot equilibrium, and positive profits in the n, cU-plane.

solution to (19) is a bit lengthy:

$$Q = U\left(1 + \frac{1}{2R(n-1)}\left(ncU - \sqrt{n^2(cU)^2 + 4n(n-1)R(cU)}\right)\right). \quad (20)$$

3.3 Positivity of Profits in the Cournot Point

Our next issue is to specify the conditions for the profits of the firms to be positive in the Cournot equilibrium for our case of identical firms. It is easiest to substitute for c_i, u_i, q_i, and Q_i directly in (11) and simplify to obtain

$$\Pi_i = \frac{1}{n}\left(R - \frac{cU^2}{U-Q}\right),$$

or, using (20),

$$\Pi_i = \frac{1}{n}\left(R - \frac{1}{2}cU - \frac{1}{2}\sqrt{c^2U^2 + 4\frac{n-1}{n}RcU}\right). \quad (21)$$

It is obvious that $\Pi_i \geq 0$ as long as

$$cU \leq \frac{n}{2n-1}R. \tag{22}$$

As $c_i u_i$ denotes the fixed cost for the individual firm, cU can be interpreted as the fixed cost of the whole industry. This must not exceed the right hand side, which depends on the number of competitors alone. It equals $\frac{2}{3}R$ in duopoly, $\frac{3}{5}R$ in triopoly, and decreases uniformly with increasing n, approaching the limit $\frac{1}{2}R$ as $n \to \infty$. The condition for nonnegative profits is hence more restrictive the larger the number of competitors. In Fig. 3 we already saw duopoly cases where profits are positive in Cournot equilibrium, as well as cases where they are not.

From (21) we also see that $\lim_{n\to\infty} \Pi_i = 0$, which means that individual profits go to zero when the number of competitors increases.

3.4 Stability of the Cournot Point

From (14), with substitutions for c_i, u_i, q_i, and Q_i from (15)-(18) and reorganizing, we obtain

$$\frac{df_i}{dQ_i} = \frac{1}{2}\sqrt{cU}\frac{R - n\left(\frac{n-1}{n}\frac{Q}{U}R\right) - 2\sqrt{\frac{n-1}{n}\frac{Q}{U}R}\sqrt{cU}}{\sqrt{\frac{n-1}{n}\frac{Q}{U}R}\left(n\sqrt{\frac{n-1}{n}\frac{Q}{U}R} + \sqrt{cU}\right)^2}, \tag{23}$$

where from (20), after a slight rearrangement,

$$\frac{n-1}{n}\frac{Q}{U}R = \frac{n-1}{n}R + \frac{1}{2}\left(cU - \sqrt{(cU)^2 + 4\frac{n-1}{n}R\,(cU)}\right). \tag{24}$$

It may be noted that (23) only depends on n, $\frac{n-1}{n}\frac{Q}{U}R$, for which we can substitute from (24), on cU, which we identified as the total fixed cost of the industry, and, of course R, which is a parameter. Further (24) depends only on n and on cU. The final expression for $\frac{df_i}{dQ_i}$, obtained through eliminating $\frac{n-1}{n}\frac{Q}{U}R$, is too long to write in one equation, but it hence depends on n and cU alone.

For our further considerations the expression $(n-1)\frac{df_i}{dQ_i}$ is crucial. This, like $\frac{df_i}{dQ_i}$ itself, ultimately depends on cU and on n alone. It is always negative, so a destabilization can only occur when

$$(n-1)\frac{df_i}{dQ_i} = -1.$$

Using this equality, with proper substitutions from (23) and (24), we get an implicit function between cU and n, which has a hyperbola shape. The lowest value of cU we can obtain is when $n \to \infty$. Therefore note that as $n \to \infty$, then $\frac{(n-1)}{n} \to 1$, and so (24) can be written

$$\frac{Q}{U}R = R + \frac{1}{2}\left(cU - \sqrt{(cU)^2 + 4R\,(cU)}\right),$$

whereas from (23)

$$\lim_{n\to\infty}(n-1)\frac{df_i}{dQ_i} = -\frac{1}{2}\frac{\sqrt{cU}}{\sqrt{\frac{Q}{U}R}}.$$

Substituting from the previous equation we get the numerical infimum value for $cU = \frac{4}{3}R$. The other asymptote is for $n = 4$, when $cU \to \infty$. This last fact implies that we can never get a destabilization of the Cournot point for any aggregate fixed cost unless the number of competitors is at least five.

Let us now write the characteristic equation. Recall that the derivatives of the reaction functions in Cournot equilibrium $\frac{df_i}{dQ_i}$ are all equal, and enter in the off diagonal elements of the Jacobian matrix. The characteristic equation is

$$\left(\lambda + \frac{df_i}{dQ_i}\right)^{n-1}\left(\lambda - (n-1)\frac{df_i}{dQ_i}\right) = 0. \tag{25}$$

Obviously there is a destabilization when $(n-1)\frac{df_i}{dQ_i} = -1$. If $n = 2$ and $\frac{df_i}{dQ_i} = -1$, there is a co-dimension 2 bifurcation with $\lambda_1 = -\frac{df_i}{dQ_i} = 1$ and $\lambda_2 = (n-1)\frac{df_i}{dQ_i} = -1$. If $n > 2$, then λ_n always bifurcates before $\lambda_1 = ...\lambda_{n-1}$ do. However, as we know, the lowest possible value of total fixed costs for destabilization is $cU = \frac{4}{3}R$ and is obtained for an infinite number of firms. This is definitely higher than the highest value for total fixed cost for which we can have positive profits, which, as we saw, stipulated $cU < kR$, where $k = \frac{n}{2n-1} \in \left[\frac{2}{3}, \frac{1}{2}\right)$, depending on the number of competitors.

Fig. 4 (where we normalized maximum consumer expenditures to unity, $R = 1$) illustrates the facts. The destabilization area in the upper right corner assumes a large number of competitors n, or a very high total fixed cost, cU. Anyhow, there is an infimum asymptote at $cU = \frac{4}{3}$. On the other hand, we see how the distance between destabilization area and the positive profit area, which is enormous for few competitors, shrinks when their number increases. But the areas stay disjoint and never intersect.

3.5 Conclusion

By conclusion, *if the Cournot equilibrium (with positive profits) exists, it cannot be destabilized in an economy with identical firms.* We can also note that whereas $\lim_{n\to\infty}(n-1)\frac{df_i}{dQ_i}$ remains negative, we always have $\lim_{n\to\infty}\frac{df_i}{dQ_i} = 0$. The fact that the firms in the limit *do not react to marginal changes of the residual supply* is an indication that we are approaching a competitive equilibrium. Finally, as we saw above, *profits are eliminated as the number of competitors goes to infinity.* Hence, everything indicates that the Cournot equilibrium seamlessly goes over into a competitive equilibrium. The capacity limits introduced, at least in the present class of cost functions, eliminate the problem of destabilization observed in previous studies due to an increasing number of competitors. This is the point of the present paper.

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