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Double knock-out Asian barrier options which widen or contract as they approach maturity

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Abstract

Barrier options are considered for Asian options using a differential equation method. Solutions are obtained in the form of Fourier series for barriers which expand or contract as they approach maturity. Rigorous bounds are obtained. It is shown that by differentiating with respect to a parameter solutions for more general payoffs can be obtained.

1   Introduction

In this paper we study the effect of barriers on exotic options. In particular barrier options are considered for Asian options using a differential equation method. Solutions are obtained in the form of Fourier series for barriers which expand or contract as they approach maturity. Rigorous bounds are obtained. It is shown that by differentiating with respect to a parameter solutions for more general payoffs can be obtained. In section 1.1 below we review previous work on barrier options and in 1.2 we describe the work developed here.

1.1   Previous Work

There are various kinds of barrier options such as plain vanilla options, time-dependent barrier options, Asian barrier options or barrier options which depend on the average of the underlying asset prices, window or limited-time barrier options. In general they are called exotic barrier options. Barrier

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options, also called trigger options, are similar to standard options except that the options are eliminated or activated when the underlying asset price reaches a predetermined barrier or boundary price. Therefore they are path-dependent options since their value depends on the price movement of the underlying asset. Barrier options are very popular in the over-the-counter market as they are more cost effective than the standard options which are path-independent. There are two main types of barrier option, the *out* option, that only pays off if a level is not reached by the asset (*knock-out* option) and the *in* option, that pays off if the level is reached by the asset before expiry (*knock-in* option). Furthermore there is the characterization of the position of the barrier relative to the position of the initial value of the underlying. If the barrier is above the initial asset value then we have an *up* option, whereas if it is under the initial asset value we have a *down* option. The payoff at expiry can be any of the common contracts, like a call, put, binary. During the life of these contracts the position of the barrier can change as time evolves either discretely or continuously. In theory barrier monitoring is assumed to be continuous, but in practice it is often discrete.

A wide variety of literature exists in the case of single barrier options. Ritchken (1995), Hui, Lo & Yuen (2000), Heritage (2002), Linetsky (1998) priced single barrier options using a wide variety of techniques. Kou (2003), Broadie, Glasserman & Kou (1999) priced discretely monitored barrier options. Lo, Yuen & Hui (2001) and Zvan, Vetzal & Forsyth (2000) used the PDE method to price single knock-out barrier options for European and discrete Asian options respectively. Roberts & Shortland (1997) priced single knock-in barriers with time-dependent coefficients. For one and multi-dimensional barriers work has been done by Firth & Dewynne (2004).

An extension of single barrier options are double barrier options. These are options which have a barrier above and below the price of the underlying asset. In this case the option gets knocked in or out as soon as one of the two barriers is hit. Kunitomo & Ikeda (1992) derived closed-form pricing formulae for curved boundaries by expressing the value of double-barrier knock-out put and call European options as an infinite series of weighted normal distribution functions. They examine the convergence of the series, by using some numerical examples and discuss a more general case of curved boundaries. Upper and lower bounds in the form of double integrals were derived by Thompson (2002) for pricing European options with a knock-out clause containing one or two curved boundaries. Schröder (2000) constructed a pricing formula series for double-barrier options which converges much faster than the one given by Kunitomo & Ikeda (1992); he provided numerical evidence by introducing a convergence parameter. Rogers & Zane (1997)

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found a method of simply transforming the knock-out moving barrier problem to a fixed barrier, which they numerically evaluate. Kolkiewicz (2002) found the values of different forms of European double barrier options with and without exponentially time dependent boundaries by representing the exit time densities as infinite series of exponential functions. Pelsser (1997), (2000) derived some pricing formulas for different kinds of basic double barrier options and expressed them in terms of trigonometric series. He used the Bromwich integral to invert the Laplace transform of the probability density function of the asset hitting the lower and upper barrier (double knock-out barrier). Hui et al. (2000) used the method of reflection to express the value of a double barrier option as the sum of infinite series of the cumulative normal distribution. Roberts & Shortland (1997) gave simple and easy-to-use method for calculating barrier options with time-dependent coefficients by applying boundary-crossing time estimation techniques in order to obtain tight bounds on the option price. Haug (1999) used a put-call transformation for valuing single barrier options to value double barrier options. Kolkiewicz (2002) provided a pricing and hedging strategy using infinite-series representations of the density functions of the exit times through the upper and lower barrier.

Most of the solutions that exist on double barrier options are either expressed as the sum of infinite series or as inverse Laplace transforms. Geman & Yor (1996) obtained a simple expression of the Laplace transform of the option price for the case of double barrier options with fixed boundaries, similar to the one they had for the Asian options and provided numerical evidence using Monte Carlo simulation. Craddock, Heath & Platen (2000) evaluated double barrier options by Laplace inverting the value of a European call option using various numerical inversion schemes and compared them. They Laplace inverted the Geman & Yor (1994) Asian Laplace transform call option solution after expressing it in terms of the confluent hypergeometric function and finally just in terms of the Gamma function. Sidenius (1998) priced double knock-out options by finding the probability density of the asset conditioned if neither the two barriers are hit during the lifetime of the option and he took the Laplace transform of the densities and found them identical to Geman and Yor. Davydov & Linetsky (2001) priced double barrier options by taking the discounted risk-neutral expectation of the payoff as the value of the option and represented the transition density as an inverse Laplace transform, as a series of normal densities and as a Fourier series. They used Euler's algorithm to invert the Laplace transform. Hui (1996) valued a one-touch double knock-out barrier binary option using the Black-Scholes framework. He provided an analytical solution to both

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European and American options using Fourier series and estimated the convergence. Under the same framework, he studied the Black-Scholes equation by pricing rear- and front-end single and double barrier call options and also the effect on the Greeks against the price and maturity, Hui (1997).

Another method to approach the valuation of a double barrier option numerically is by using a partial differential equation. Zvan et al. (2000) valued double barrier European options using a fully implicit method to solve the PDE numerically and compared it with the Crank-Nicolson method. They also considered the case of a two-asset barrier option and produced numerical evidence. Linetsky (2002) studied barrier and arithmetic Asian options by using a PDE and taking eigenfunction expansions. Similarly, Davydov & Linetsky (2003) priced single and double barrier options using the linearity of the pricing operator and the eigenvector property of the eigen-securities and illustrated them numerically using the constant elasticity of variance process and the Cox-Ingersoll-Ross term-structure model. Broadie et al. (1999) used the pricing formula for continuous path-dependent options to approximate the price of discrete options and used the trinomial lattice method to price discrete barrier options.

An interesting effect is the presence of transaction costs when valuing options. Leland (1985) considered a model that allows transactions only at discrete times. By adopting a δ-hedging argument, he derived an option price that converges to a Black-Scholes price as transaction costs become arbitrarily small with an adjusted volatility. Barles & Soner (1998) applied a utility function approach as well as an asymptotic analysis of a nonlinear PDE under the effect of transaction costs. Gondzio, Kouwenberg & Vorst (2003) proposed a stochastic optimization model for hedging contingent claims and take into account the effects of stochastic volatility and transaction costs. The presence of transaction costs affect the volatility and therefore the governing equation. Morozovsky (2000) constructed a modified Black-Scholes equation with transaction costs and gave a solution to the equation for the case where the option was close to expiry and the case where the transaction costs were small. Dewynne, Whalley & Wilmott (1994) constructed a PDE model for valuing exotic options with the possibility of a similarity solution. They used explicit finite differences to overcome the strong nonlinearity caused by the transaction costs term.

1.2 Plan of paper

The type of barrier that we will examine is the double barrier, where the two barriers are positioned under and above the initial value of the asset.

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We will study the effect of double knock-out barriers for European Asian options which can move either widening or narrowing. The payoff of such an option is based on the history of the price of the underlying asset up to the expiration date.

We consider the problem of European Asian options with barriers (section 2.1) and in addition barriers which can move either widening (section 2.2) or narrowing (section 2.3) as we approach maturity. We begin by formulating the problem in terms of differential equations which depend on both the underlying asset S and a running sum $I = \int_0^t S_\tau \, d\tau$ (section 2). We use a particular substitution of the form $V(S, I(t), t) = \exp{(\alpha I(t))} F(S, t)$ and hence reduce the underlying equations to a simpler form involving equations of a form including both S and t as independent variables. Note that for barriers which are knock-out the condition $V = 0$ on the barrier becomes $F = 0$. However to be consistent the payoff $V(S, I(T), T)$ must have a particular form i.e. $V(S, I(T), T) = \exp{(\alpha I(T))} F(S, T)$.

Once this class of solutions has been found (here we use a combination of the WKB asymptotic solutions of the underlying partial differential equations as well as Fourier series) we can generalise the solution to more general payoffs by differentiation with respect to the parameter α. These solutions are sufficiently accurate, however exact solutions in terms of Bessel functions can also be used (section B).

In section 2 we derive the general equations for the problem along with the transformation $V(S, I(t), t) = \exp{(\alpha I(t))} F(S, t)$ used to reduce the dimensionality of the problem.

In sections 2.1-2.3 we describe the nature of the barrier for three problems along with the equations and their solutions: a) a fixed barrier to maturity (see figure 1), b) an expanding barrier (see figure 2) and c) a narrowing barrier (figure 3).

These problems are generalised by differentiating the value of the option with respect to a parameter (section 3) and hence deducing results for more general payoffs.

Finally in section 4 results are presented along with bounds for the value of each of the barrier options. We discuss how the price is affected by the two barriers, the jumps and the time to maturity.

A brief study of Asian options with transaction costs is covered in the appendix, section A. The problem is approached using a one-dimensional PDE involving a similarity variable and time t. A method is introduced in determining the value of the Greek gamma for single signed payoffs. The results given there could be taken to apply strictly to the limit case of a barrier option with one barrier coinciding with zero exercise price ($E = 0$)

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and an upper barrier tending to infinity. More work is required to use these results for the finite barrier problem.

2 General equations

In general, the value of an average value option depends on the average value of the stock I, the value of the stock S as well as on the time t, $V(S, I, t)$.

We first derive the general equations for $V(S, I, t)$ but then simplify the problems by looking for solutions of the form

$$V(\bar{S}, \bar{I}, t) = \exp(\alpha_1 \bar{I}(t)) F(\bar{S}, t). \tag{1}$$

Later on we use a general payoff to find more general solutions to the problem. This is accomplished by differentiation with respect to the parameter α_1.

For a small time-step $t \to t + dt$, the value of the option changes by

$$dV(S, I, t) = \left(\frac{\partial V}{\partial t} + S \frac{\partial V}{\partial I} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \frac{\partial V}{\partial S} dS,$$

using the properties of the Brownian motion, where $I = \int_0^t S(\tau) \, d\tau$ is the running sum. We construct a portfolio Π which changes by

$$d\Pi = \left(\frac{\partial V}{\partial t} + S \frac{\partial V}{\partial I} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \frac{\partial V}{\partial S} dS - \Delta \, dS.$$

The risk is eliminated by choosing $\Delta = \frac{\partial V}{\partial S}$, and the change in the portfolio's value is the same as the growth of the equivalent amount of cash in a risk-free interest-bearing bank account. This results in a partial differential equation

$$\frac{\partial V}{\partial t} + S \frac{\partial V}{\partial I} + rS \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0. \tag{2}$$

We eliminate the running sum I by using (1), where $\bar{S} = \frac{S}{S_0}$, $\bar{I} = \frac{I}{S_0}$ and $\alpha_1 = \alpha S_0$; S_0 being the initial asset value.

The above equation reduces to one involving two independent variables, $\bar{S}$ and t,

$$\frac{\partial F}{\partial t} + \frac{1}{2} \sigma^2 \bar{S}^2 \frac{\partial^2 F}{\partial \bar{S}^2} + r\bar{S} \frac{\partial F}{\partial \bar{S}} + (\alpha_1 \bar{S} - r)F = 0. \tag{3}$$

Later on we use a general payoff to find more general solutions to the problem. Note that throughout the paper $\bar{S} = S/S_0$, $\bar{S}_L = S_L/S_0$, $\bar{S}_U = S_U/S_0$, $\bar{S}_{WL} = S_{WL}/S_0$, $\bar{S}_{WU} = S_{WU}/S_0$, $\bar{S}_{SL} = S_{SL}/S_0$ and $\bar{S}_{SU} = S_{SU}/S_0$.

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2.1   The fixed barrier problem (figure 1)

Here two barriers are considered which are parallel to each other remaining constant throughout the life of the option. At time $t = T$ the two barriers are positioned at levels $S = S_L$ and $S = S_U$, with $S_L < S < S_U$, letting the underlying move freely in the area created between the boundaries, figure 1. The two barriers are of knock-out nature, meaning that the option will
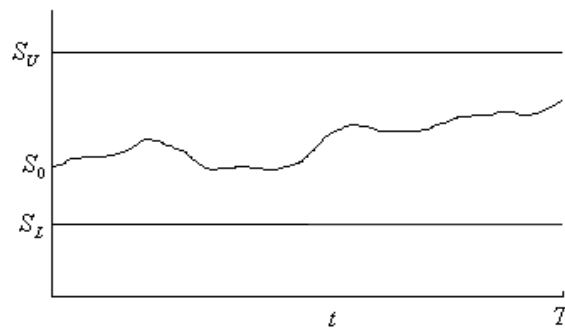

Figure 1: Path of an asset with the effect of double barriers.

expire worthless if one of the two barriers is triggered. The value is governed by equation (2) subject to the boundary conditions

$$V(S, I, t) = 0, \quad \text{at} \quad S = S_L, \; S = S_U, \tag{4}$$

where S_L is the lower barrier, S_U is the upper barrier and $S_L < S < S_U$.

From the substitution (1) the condition $V(S, I, t) = 0$ on the two barriers will become $F(\bar{S}, t) = 0$ with (3) as the corresponding equation and conditions at the two boundaries

$$F(\bar{S}, t) = 0, \quad \text{at} \quad \bar{S} = \bar{S}_L, \; \bar{S} = \bar{S}_U, \tag{5}$$

where $\bar{S}_L = \frac{S_L}{S_0}, \bar{S}_U = \frac{S_U}{S_0}$ with $0 < \bar{S}_L < 1$ and $\bar{S}_U > 1$.

Equation (3) can be reduced to the following equation

$$\frac{\partial W}{\partial t_2} = \frac{\partial^2 W}{\partial \bar{x}^2} + \alpha_2 \exp(\bar{x}) W, \tag{6}$$

using the transformations

$$t_1 = T - t, \quad \bar{S} = \exp(\bar{x}), \quad F(\bar{x}, t_1) = \exp(\beta t_1 + \gamma \bar{x}) W(\bar{x}, t_1),$$

$$\beta = -\frac{\sigma^2}{8} - \frac{r}{2} - \frac{r^2}{2\sigma^2}, \quad \gamma = \frac{1}{2} - \frac{r}{\sigma^2}, \quad t_2 = \frac{\sigma^2}{2} t_1, \quad \alpha_2 = \frac{2\alpha_1}{\sigma^2}.$$

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The boundary conditions for the problem are now

$$W(\bar{x}, t_2) = 0, \quad \text{at} \quad \bar{x} = \bar{x}_L \quad \text{and} \quad \bar{x} = \bar{x}_U. \tag{7}$$

Moving the origin, equation (6) becomes

$$\frac{\partial W}{\partial t_2} = \frac{\partial^2 W}{\partial \tilde{x}^2} + \alpha_3 \exp(\tilde{x})W, \tag{8}$$

where $\tilde{x} = \bar{x} - \bar{x}_L$, $\alpha_3 = \alpha_2 \exp(\bar{x}_L)$ and the boundary conditions at the two barriers take the form

$$W(0, t_2) = W(\bar{x}_U - \bar{x}_L, t_2) = 0. \tag{9}$$

Equation (8) is solved using separation of variables. The solution can be written in terms of Bessel functions or can be approximated using a combination of the WKB method (named after Wentzel, Kramers and Brillouin), e.g. Bender & Orzsag (1978), as well as Fourier series,

$$W(\tilde{x}, t_2) = \sum_{n=1}^{\infty} A_n \exp(-k_n^2 t_2)(\alpha_3 \exp(\tilde{x}) + k_n^2)^{-1/4}$$
$$\sin\left(\int_0^{\tilde{x}} \sqrt{\alpha_3 \exp(\tau) + k_n^2}\, d\tau\right), \tag{10}$$

satisfying the boundary condition at $\tilde{x} = 0$. k_n are the eigenvalues given by applying the other boundary condition at $\tilde{x} = \bar{x}_U - \bar{x}_L$ which requires that

$$\int_0^{\bar{x}_U - \bar{x}_L} \sqrt{\alpha_3 \exp(\tau) + k_n^2}\, d\tau = n\pi. \tag{11}$$

The coefficients A_n will be determined from the final (payoff) condition on the problem (i.e. the initial conditions have at $t_2 = 0$), see (26).

Note that the expansion (10) implicitly assumes that (11) has a positive solution for k_n^2 for $n = 1$. This is the case in the examples considered later. But see the discussion in (section 3.2) for cases where the width of the barrier is such that this is not the case. The summation then is simply replaced by $n = n^*$ to infinity where n^* is determined (see section 3.1).

2.2 When the barriers widen (figure 2)

There are two constant barriers $\bar{S}_{WL}$ and $\bar{S}_{WU}$ which at some time $t = t^* < T$ change symmetrically up and down to take the corresponding values $\bar{S}_L$,

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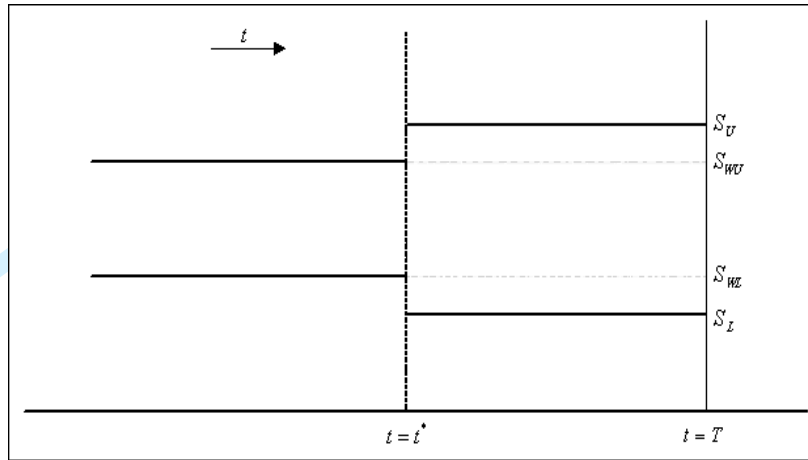
Figure 2: Jump of the barrier at $t = t^*$.

$\bar{S}_U$ till expiry, with $\bar{S}_L < \bar{S}_{WL} < \bar{S}_{WU} < \bar{S}_U$. This change results in a wider area between the two barriers $\bar{S}_L$, $\bar{S}_U$ leaving more space for the underlying to move freely (figure 2).

For this problem we have the same equation as mentioned earlier, (6), with the option value being zero at the barriers $\bar{S} = \bar{S}_{WL}$ and $\bar{S} = \bar{S}_{WU}$ for $t < t^*$. The origin is moved by setting $\hat{x} = \bar{x} - \bar{x}_{WL}$ and the equation takes the form

$$\frac{\partial W_W}{\partial t_2} = \frac{\partial^2 W_W}{\partial \hat{x}^2} + \alpha_4 \exp(\hat{x}) W_W \,, \tag{12}$$

where $\alpha_4 = \alpha_2 \exp(\bar{x}_{WL})$ and with boundary conditions

$$W_W(0, t_2) = W_W(\bar{x}_{WU} - \bar{x}_{WL}, t_2) = 0 \,. \tag{13}$$

Note that W_W is the value of the option when the barrier widens.

The solution to equation (12) is given by

$$W_W(\hat{x}, t_2) \;=\; \sum_{n=1}^{\infty} C_n \exp\left(-\lambda_n^2 (t_2 - t_2^*)\right) \tag{14}$$

$$(\alpha_4 \exp(\hat{x}) + \lambda_n^2)^{-1/4} \sin\left(\int_0^{\hat{x}} \sqrt{\alpha_4 \exp(\tau) + \lambda_n^2}\, d\tau\right) \,,$$

valid for $t_2 < t_2^*$, where λ_n are the eigenvalues given by applying the other boundary condition at $\hat{x} = \bar{x}_{WU} - \bar{x}_{WL}$ which requires that

$$\int_0^{\bar{x}_{WU} - \bar{x}_{WL}} \sqrt{\alpha_4 \exp(\tau) + \lambda_n^2}\, d\tau = n\pi \,. \tag{15}$$

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The coefficients C_n are determined by conditions at $t_2 = t_2^*$ which follow from equation (10).

2.3 When the barriers narrow (figure 3)

For this problem at time $t = t^* < T$ two barriers, $\bar{S}_{SL}$ and $\bar{S}_{SU}$ move symmetrically inwards to values $\bar{S}_L$, $\bar{S}_U$, with $\bar{S}_{SL} < \bar{S}_L < \bar{S}_U < \bar{S}_{SU}$, making the area between the barriers smaller than before (figure 3).
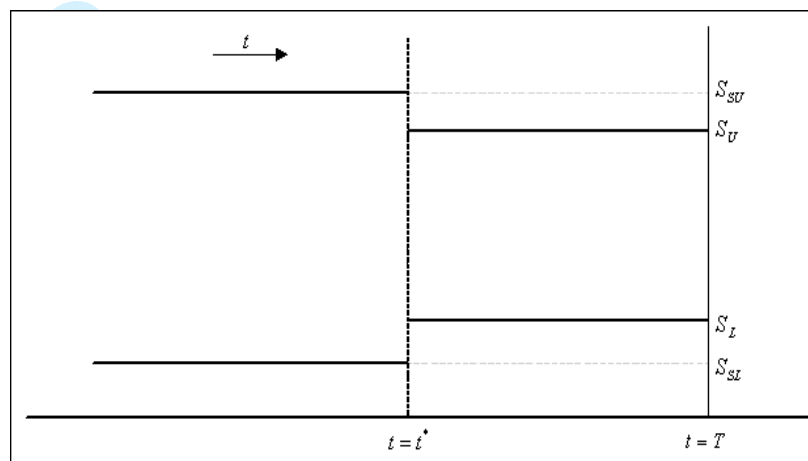
Figure 3: Jump of the barrier at $t = t^*$.

The solution for this problem is given by the equation

$$
\begin{aligned}
W_S(\check{x}, t_2) \;=\; & \sum_{n=1}^{\infty} E_n \exp\left(-\xi_n^2 (t_2 - t_2^*)\right) \\
& \left(\alpha_5 \exp(\check{x}) + \xi_n^2\right)^{-1/4} \sin\left(\int_0^{\check{x}} \sqrt{\alpha_5 \exp(\tau) + \xi_n^2}\, d\tau\right),
\end{aligned} \tag{16}
$$

valid for $t_2 < t_2^*$, where W_S is the value of the option when the barrier narrows, $\check{x} = \bar{x} - \bar{x}_{SL}$, $\alpha_5 = \alpha_2 \exp(\bar{x}_{SL})$ and ξ_n is given by the equation

$$
\int_0^{\bar{x}_{SU} - \bar{x}_{SL}} \sqrt{\alpha_5 \exp(\tau) + \xi_n^2}\, d\tau = n\pi. \tag{17}
$$

The coefficients E_n are formed from conditions at $t_2 = t_2^*$ which are found by extracting the solution of the form given in equation (10) such that in the regions beyond the smaller boundaries at $t_2 = t_2^*$ the option price is zero. Note that $t_2 = t_2^*$ means $t = t^*$ in the figure.

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3 Determining the coefficients in the expansions and some generalisation

For ordinary European Asian options the payoff for a call option is given by the equation

$$V_C(S, I, T) = \max\left(\frac{I}{T} - E, 0\right).$$

For European Asians with barriers we construct a payoff of the form

$$V^*(S, I, T) = \exp(\alpha I)\left(\frac{I}{T} - S_L\right),\tag{18}$$

which is similar to the ordinary value of an Asian call option. In general the strike price, E, is situated between the two barriers at maturity $t = T$. Here the value of the strike E is taken to be equal to the value of the lower barrier S_L. Note that for α tending to zero, this is a barrier option with the usual European option payoff since $\max\left(\frac{I}{T} - S_L, 0\right) = \frac{I}{T} - S_L$ in this case with $E = S_L$. At any time t the running sum $I(t)$ will take values between $S_L t < I(t) < S_U t$, where $S_L t$ and $S_U t$ are the minimum and maximum values the running sum takes at the two barriers and $I(t) = \int_0^t S_u \, du$ (see figure 4).
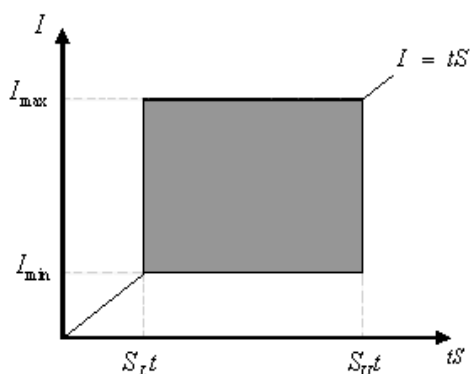

Figure 4: Shaded area where the sum takes value at time t.

The payoff (18) can be thought of as corresponding to two options. One where the function $F(S, t)$ is equal to $F(S, t) = S_L = E$ so that the payoff for that option is

$$V^{(1)}(S, I, T) = S_L \exp(\alpha I).\tag{19}$$

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To find the price of another option $V^{(2)}(S,I,T)$ we first consider an option where the function $F(S,t)$ is equal to $F(S,t) = \frac{1}{T}$ with payoff

$$V^{**}(S,I,T) = \frac{1}{T} \exp(\alpha I), \tag{20}$$

and we differentiate this solution with respect to α to get an option with payoff

$$V^{(2)}(S,I,T) = \frac{\partial V^{**}}{\partial \alpha} = \frac{I}{T} \exp(\alpha I). \tag{21}$$

$V^{(1)}(S,I,T)$ and $V^{(2)}(S,I,T)$ are considered as two different problems and combining the two payoffs results in the original payoff,

$$V^{(2)}(S,I,T) - V^{(1)}(S,I,T) = \exp(\alpha I)\left(\frac{I}{T} - S_L\right). \tag{22}$$

3.1 The eigenvalues of the expansions (10), (14), (16)

Note that in determining the coefficients of the expansions (10), (14), (16) we first need to solve the equations such as (11) for the eigenvalues k_n, λ_n, ξ_n. The existence of these eigenvalues depends on the width $\bar{x}_U - \bar{x}_L$. In the examples we consider later this value is sufficiently small so that $k_1 > 0$ exists. A necessary condition for this is that

$$\int_0^{\bar{x}_U - \bar{x}_L} \sqrt{\alpha_3 \exp(\tau)}\, d\tau < \pi, \tag{23}$$

i.e. $2\sqrt{\alpha_3}(\exp((\bar{x}_U - \bar{x}_L)/2 - 1)) < \pi$.

The same argument applies to (14) and (16). Clearly as $\bar{x}_U - \bar{x}_L$ increases we must replace the summations from $n = n^*$ to infinity.

In general the critical value of n^* is determined by the inequality

$$\ln\left(\frac{(n^* - 1)\pi}{2\sqrt{\alpha_3}} + 1\right) < \frac{1}{2}(\bar{x}_U - \bar{x}_L) < \ln\left(\frac{n^*\pi}{2\sqrt{\alpha_3}} + 1\right). \tag{24}$$

Note that for $\frac{1}{2}(\bar{x}_U - \bar{x}_L) \gg 1$, then n^* is $\gg 1$ and successive values of n are very close together so that the summation can be replaced by an integral.

It is of course well known that finite barriers give rise to a discrete set of eigenvalues whereas the infinite domain has a continuous set. Since our prime purpose here is the barrier problem itself this limit although an interesting analytical exercise is not very informative. To treat this limit the natural way to do the calculation would be to Laplace transform equation (3)

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over time and solve the resulting ordinary differential equation either with barriers or in an infinite domain with some final conditions. This resulting ordinary differential equation has a complicated solution in terms of Bessel functions, etc. For the barrier problem inverting the transform will result in an infinite set of poles (directly related to our Fourier series solution when these residues are evaluated). The infinite domain problem however will have branch cuts in the complex plane when the transform is inverted. Such a connection is natural for equations such as equation (3), although while clear as for the equation is concerned the resulting details will of course not be necessarily transparent.

It is clear from the above discussion that as $\bar{x}_U - \bar{x}_L$ tends to infinity the integral approximation to the sum becomes increasingly accurate. The practical difficulty with the transformed version of the problem versus the series solution we use here is when the barriers shrink or grow since here we have a new initial condition for the Laplace transforms method.

3.2 The barrier option

For the final payoff we take a linear function of the asset $\bar{S}$ forming a triangle between the two barriers $\bar{S}_L$ and $\bar{S}_U$ (straight solid line in figure 5) of the form $V(\bar{S}, \bar{I}, T) = \exp(\alpha_1 \bar{I}) F(\bar{S}, t)$, where

$$F(\bar{S}, t) = \begin{cases} \frac{\bar{S}_U - \bar{S}}{\bar{S}_U - \bar{S}_L} & , \frac{\bar{S}_U + \bar{S}_L}{2} < \bar{S} < \bar{S}_U \\ \frac{\bar{S} - \bar{S}_L}{\bar{S}_U - \bar{S}_L} & , \bar{S}_L < \bar{S} < \frac{\bar{S}_U + \bar{S}_L}{2} \end{cases} , \tag{25}$$

which in terms of the variable $\tilde{x}$ takes the form,

$$F(\tilde{x}, 0) = \begin{cases} \frac{\bar{S}_U - \bar{S}_L \exp(\tilde{x})}{\bar{S}_U - \bar{S}_L} & , \ln\left(\frac{\bar{S}_U + \bar{S}_L}{2\bar{S}_L}\right) < \tilde{x} < \ln\left(\frac{\bar{S}_U}{\bar{S}_L}\right) \\ \frac{\bar{S}_L \exp(\tilde{x}) - \bar{S}_L}{\bar{S}_U - \bar{S}_L} & , 0 < \tilde{x} < \ln\left(\frac{\bar{S}_U + \bar{S}_L}{2\bar{S}_L}\right) \end{cases} . \tag{26}$$

3.2.1 The fixed barrier problem

The full solution for the double barrier Asian option discussed in section 2.1 is

$$\begin{aligned} V(\bar{S}, \bar{I}, t) &= \exp(\alpha_1 \bar{I}) \exp\left(-\left(\frac{r^2}{2\sigma^2} + \frac{\sigma^2}{8} + \frac{r}{2}\right)(T-t)\right) \bar{S}^{\left(\frac{1}{2} - \frac{r}{\sigma^2}\right)} \\ &\quad \sum_{n=1}^{\infty} A_n \exp\left(-\frac{\sigma^2}{2} k_n^2 (T-t)\right) \left(\frac{2\alpha_1}{\sigma^2} \bar{S} + k_n^2\right)^{-1/4} \\ &\quad \sin\left(\int_1^{\bar{S}/\bar{S}_L} \sqrt{\frac{2\alpha_1}{\sigma^2} \bar{S}_L v + k_n^2} \frac{dv}{v}\right), \end{aligned} \tag{27}$$

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for $0 < t < T$, with $V(\bar{S}, \bar{I}, t) = 0$ at $\bar{S} = \bar{S}_L$ and $\bar{S} = \bar{S}_U$, where

$$A_n = \frac{\int_0^{\bar{x}_U - \bar{x}_L} W(\tilde{x}, 0) W_n(\tilde{x}) \, d\tilde{x}}{\int_0^{\bar{x}_U - \bar{x}_L} [W_n(\tilde{x})]^2 \, d\tilde{x}} \, . \tag{28}$$

Note that $W(\tilde{x}, 0)$ involves the payoff at $t_2 = 0$ ($t = T$).

3.2.2 The widening barrier problem

For the problem where the two barriers widen at time $t = t^*$, the full solution is given by the equation

$$
\begin{aligned}
V_W(\bar{S}, \bar{I}, t) &= \exp(\alpha_1 \bar{I}) \exp\left(-\left(\frac{r^2}{2\sigma^2} + \frac{\sigma^2}{8} + \frac{r}{2}\right)(T-t)\right) \bar{S}^{\left(\frac{1}{2} - \frac{r}{\sigma^2}\right)} \\
&\quad \sum_{n=1}^{\infty} C_n \exp\left(-\frac{\sigma^2}{2} \lambda_n^2 (t^* - t)\right) \left(\frac{2\alpha_1}{\sigma^2} \bar{S} + \lambda_n^2\right)^{-1/4} \\
&\quad \sin\left(\int_1^{\bar{S}/\bar{S}_{WL}} \sqrt{\frac{2\alpha_1}{\sigma^2} \bar{S}_{WL} v + \lambda_n^2} \, \frac{dv}{v}\right),
\end{aligned}
\tag{29}
$$

valid for all times $0 < t < t^*$ and at $\bar{S} = \bar{S}_{WL}$, $\bar{S} = \bar{S}_{WU}$, $V_W(\bar{S}, \bar{I}, t) = 0$, where

$$C_n = \frac{\int_0^{\bar{x}_{WU} - \bar{x}_{WL}} W(\hat{x}, t_2^*) W_{Wn}(\hat{x}) \, d\hat{x}}{\int_0^{\bar{x}_{WU} - \bar{x}_{WL}} [W_{Wn}(\hat{x})]^2 \, d\hat{x}} \, . \tag{30}$$

Note that the condition at $t_2 = t_2^*$ ($t = t^*$) is formed from the previous solution.

3.2.3 The narrowing barrier problem

In the case where the two barriers narrow at time $t = t^*$, the value of the option will look like

$$
\begin{aligned}
V_S(\bar{S}, \bar{I}, t) &= \exp(\alpha_1 \bar{I}) \exp\left(-\left(\frac{r^2}{2\sigma^2} + \frac{\sigma^2}{8} + \frac{r}{2}\right)(T-t)\right) \bar{S}^{\left(\frac{1}{2} - \frac{r}{\sigma^2}\right)} \\
&\quad \sum_{n=1}^{\infty} E_n \exp\left(-\frac{\sigma^2}{2} \xi_n^2 (t^* - t)\right) \left(\frac{2\alpha_1}{\sigma^2} \bar{S} + \xi_n^2\right)^{-1/4} \\
&\quad \sin\left(\int_1^{\bar{S}/\bar{S}_{SL}} \sqrt{\frac{2\alpha_1}{\sigma^2} \bar{S}_{SL} v + \xi_n^2} \, \frac{dv}{v}\right),
\end{aligned}
\tag{31}
$$

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valid for all times $0 < t < t^*$ and at $\bar{S} = \bar{S}_{SL}$, $\bar{S} = \bar{S}_{SU}$, $V_S(\bar{S}, \bar{I}, t) = 0$, where

$$E_n = \frac{\int_{\bar{x}_L - \bar{x}_{SL}}^{\bar{x}_U - \bar{x}_{SL}} W(\check{x}, t_2^*) W_{Sn}(\check{x}) \, d\check{x}}{\int_0^{\bar{x}_{SU} - \bar{x}_{SL}} [W_{Sn}(\check{x})]^2 \, d\check{x}} \, . \tag{32}$$

Note that the condition at $t_2 = t_2^*$ ($t = t^*$) is formed from the solution (27).

3.3   Other payoffs

Going back to the generalised problem referred to in section 3, the solution to the option with payoff (19) in terms of the variables $\bar{S}$, $\bar{I}$ and t will be given by equation (27) with coefficient $\bar{S}_L$,

$$
\begin{aligned}
V^{(1)}(\bar{S}, \bar{I}, t) \;=\; & \bar{S}_L \exp\left(\alpha_1 \bar{I}\right) \exp\left(-\left(\frac{r^2}{2\sigma^2} + \frac{\sigma^2}{8} + \frac{r}{2}\right)(T - t)\right) \bar{S}^{\left(\frac{1}{2} - \frac{r}{\sigma^2}\right)} \\
& \sum_{n=1}^{\infty} A_n \exp\left(-\frac{\sigma^2}{2} k_n^2 (T - t)\right) \left(\frac{2\alpha_1}{\sigma^2} \bar{S} + k_n^2\right)^{-1/4} \\
& \sin\left(\int_1^{\bar{S}/\bar{S}_L} \sqrt{\frac{2\alpha_1}{\sigma^2} \bar{S}_L \, v + k_n^2} \, \frac{dv}{v}\right),
\end{aligned}
\tag{33}
$$

with coefficients given by the equation

$$A_n = \frac{\int_{\bar{S}_L}^{\bar{S}_U} W(\bar{S}, T) W_n(\bar{S}) \, d\bar{S}}{\int_{\bar{S}_L}^{\bar{S}_U} [W_n(\bar{S})]^2 \, d\bar{S}} \, , \tag{34}$$

with $W(\bar{S}, T) = F(\bar{S}) = 1$.

For the option with payoff (20) the solution is similarly given by

$$
\begin{aligned}
V(\bar{S}, \bar{I}, t) \;=\; & \frac{1}{T} \exp\left(\alpha_1 \bar{I}\right) \exp\left(-\left(\frac{r^2}{2\sigma^2} + \frac{\sigma^2}{8} + \frac{r}{2}\right)(T - t)\right) \bar{S}^{\left(\frac{1}{2} - \frac{r}{\sigma^2}\right)} \\
& \sum_{n=1}^{\infty} A_n \exp\left(-\frac{\sigma^2}{2} k_n^2 (T - t)\right) \left(\frac{2\alpha_1}{\sigma^2} \bar{S} + k_n^2\right)^{-1/4} \\
& \sin\left(\int_1^{\bar{S}/\bar{S}_L} \sqrt{\frac{2\alpha_1}{\sigma^2} \bar{S}_L \, v + k_n^2} \, \frac{dv}{v}\right),
\end{aligned}
$$

with the same coefficients, A_n, as in the former option, (33). Differentiating the latter with respect to α_1 and subtracting $V^{(1)}(\bar{S}, \bar{I}, t)$ will give the value of the double barrier Asian option,

$$V(\bar{S}, \bar{I}, t) = \frac{dV}{d\alpha}(\bar{S}, \bar{I}, t) - V^{(1)}(\bar{S}, \bar{I}, t) \, , \tag{35}$$
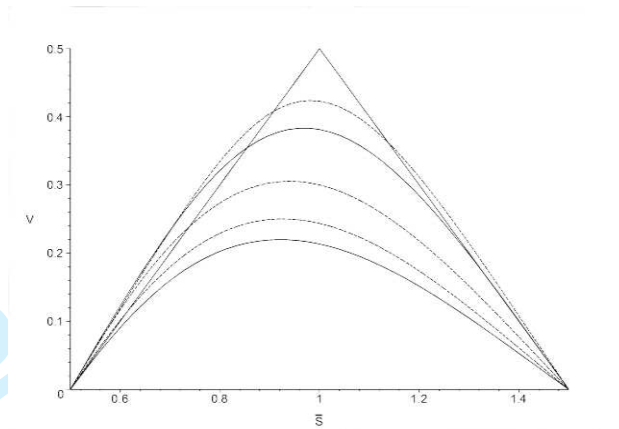
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Figure 5: Plot of the solution $V(\bar{S}, \bar{I}, t)$, (27), for $\bar{I} = 0$ at times $t = 1$ (payoff, straight solid line) $t = 0.9$ (top dashed line), $t = 0.8$ (top solid line), $t = 0.5$ (middle dashed line), $t = 0.2$ (bottom dashed line) and $t = 0$ (bottom solid line).

with payoff (18). This can be seen more analytically in section C.

We can apply the method introduced in the previous sections (2.2 and 2.3) for different payoffs, in the case where the two barriers jump inwards and outwards. For this problem bounds can be found in section (4.1.1).

4   Results

The plots of the above solutions (the fixed barrier, (27), the widening barrier, (29) and the narrowing barrier (31)) with payoff (25) can be viewed in figures 5, 6 and 7 and with $\bar{I}$ being zero. For the numerical example we have taken $\bar{S}_L = 0.5$, $\bar{S}_U = 1.5$, $\bar{S}_{WL} = 0.7$, $\bar{S}_{WU} = 1.3$, $\bar{S}_{SL} = 0.3$, $\bar{S}_{SU} = 1.7$, $\alpha_1 = 0.5$, $\sigma = 0.5$, $r = 0.05$ and with maturity $T = 1$ year. At any time, the running sum $\bar{I}$ takes values between $\bar{S}_L t < \bar{I}(t) < \bar{S}_U t$.

In figure 5 as time moves away from expiry (straight solid line), the value of the option drops and becomes cheaper (other lines), as the chances of the underlying hitting one of the barriers increases.

In figure 6 is the plot of the solution $V_W(\bar{S}, \bar{I}, t)$ for different times. At $t = 0$ the value of the option is cheap as it is far away from the maturity. At the jump $t = t^*$ the value of the option will become more expensive as the holder of the option has benefited from this sudden change of luck after managing to avoid hitting any of the barriers. For $t > t^*$ the possibilities
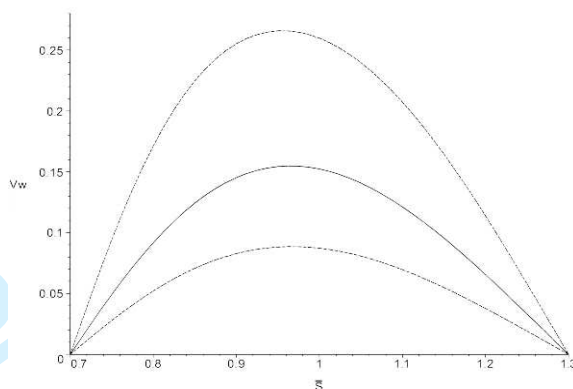
16

Figure 6: Plot of the solution $V_W(\bar{S}, \bar{I}, t)$, (29), for $\bar{I} = 0$ at times $t = 0.4$ (top dashed line), $t = 0.2$ (solid line) and $t = 0$ (bottom dashed line).

of the underlying hitting one of the new barriers have decreased and as the time to maturity approaches, the value of the option increases to reach a value at the exercise date.

For the value of the option as time tends to $t = t^*$, just before the barrier widens, i.e. forward in real time, say at time $t^* - \epsilon$ ($\epsilon \ll 1$), the option has some value (it's not zero as it hasn't hit any barrier) different to the one occurring at time $t^* + \epsilon$. At $t^* - \epsilon < t < t^* + \epsilon$ the function is discontinuous, since it only takes values for $\bar{S}_{WL} < \hat{x} < \bar{S}_{WU}$ and zero for values $\bar{S}_L < \hat{x} < \bar{S}_{WL}$, $\bar{S}_{WU} < \hat{x} < \bar{S}_U$. The holder of the option at $t^* + \epsilon$ is in much better position than the holder at $t^* - \epsilon$, because he has less chances of hitting the barriers. As a consequence, the value of the option increases from position $t^* - \epsilon$, before the jump, to the position $t^* + \epsilon$, after the jump.

Figure 7 shows the value of the option when the two barriers shrink. Before the jump occurs at $t = t^*$ the holder of the option is in a better position than after as the underlying has more space to move and is less likely to hit the barrier $[\bar{S}_{SL}, \bar{S}_{SU}]$. If the underlying is close to one of the two barriers at $t^* - \epsilon$, the asset will not have time to manoeuvre in order to fit into the new barrier and the value of the option will drop. This is because the underlying is more likely to miss the shrunken area and therefore expire worthless.

In figure 8 we plot the values of the three barrier options (equations (27)-(31)) at present time, i.e. $t = 0$ ($\bar{I} = 0$ at that time). Looking at these graphs we can draw the following conclusions for the value of the
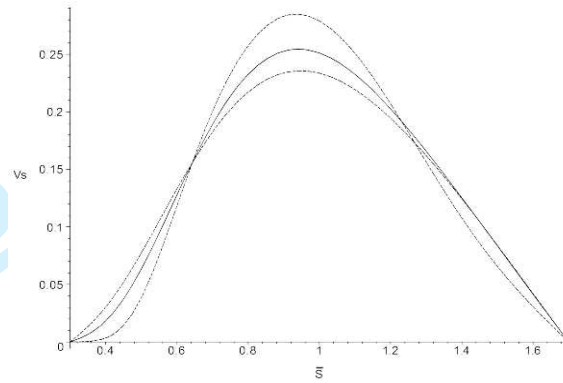
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Figure 7: Plot of the solution $V_S(\bar{S}, \bar{I}, t)$, (31), for $\bar{I} = 0$ at times $t = 0.4$ (top dashed line), $t = 0.2$ (solid line) and $t = 0$ (bottom dashed line).
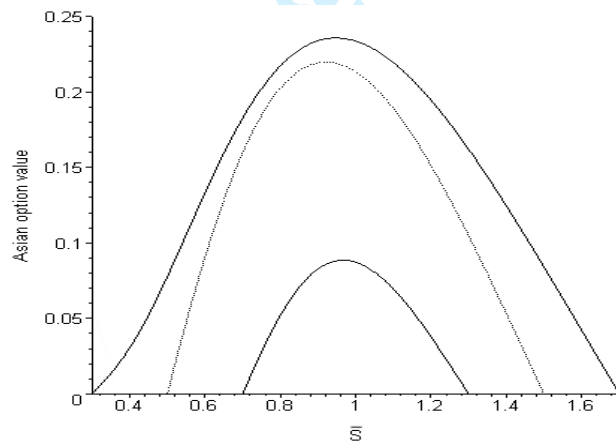

Figure 8: Plot of the solutions $V(\bar{S}, \bar{I}, t)$ (dashed curve), $V_W(\bar{S}, \bar{I}, t)$ (lower solid curve), $V_S(\bar{S}, \bar{I}, t)$ (upper solid curve) for $\bar{I} = 0$ at time $t = 0$.
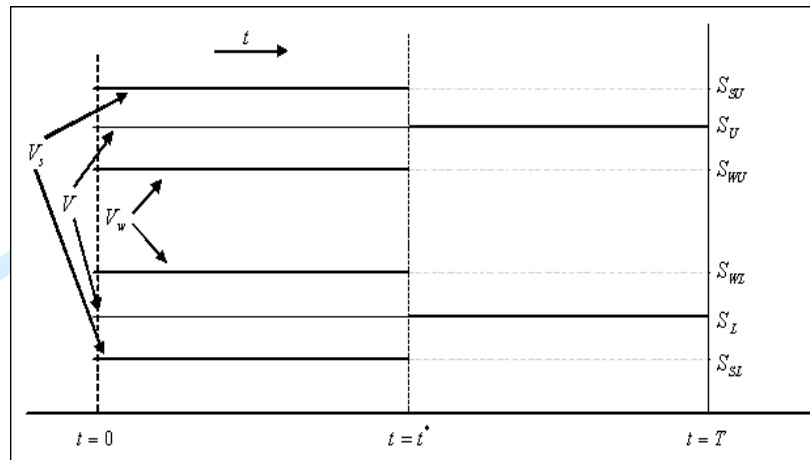
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Figure 9: Sketch of the three types of barriers.

option at time $t = 0$. The value of the three options depends on the size of the barriers. More specifically, the value of the simple barrier at $t = 0$, $[\bar{S}_L, \bar{S}_U]$, is cheaper than the value of the option when the barriers jump inward, $[\bar{S}_{SL}, \bar{S}_{SU}]$, at that time (figure 9). This is because the holder of the option enters the contract having a better position since the two barriers, $[\bar{S}_{SL}, \bar{S}_{SU}]$, are further apart. The value of the barrier which jumps outwards will be much cheaper than the value of the original barrier, $[\bar{S}_L, \bar{S}_U]$, at that time ($t = 0$), as the underlying has more chances of hitting either barrier $[\bar{S}_{WL}, \bar{S}_{WU}]$.

4.1 Bounds

Applying the same payoff for the case of European options we can get bounds for the value of the Asian option at $t = 0$. Consider a European option with payoff between values $\exp(\alpha_1 \bar{I}_{min})V_{BS}$ and $\exp(\alpha_1 \bar{I}_{max})V_{BS}$, where V_{BS} is the value of the Black-Scholes equation with payoff

$$V_{BS}(\bar{S}, T) = \begin{cases} \frac{\bar{S}_U - \bar{S}}{\bar{S}_U - \bar{S}_L} & , \frac{\bar{S}_U + \bar{S}_L}{2} < \bar{S} < \bar{S}_U \\ \\ \frac{\bar{S} - \bar{S}_L}{\bar{S}_U - \bar{S}_L} & , \bar{S}_L < \bar{S} < \frac{\bar{S}_U + \bar{S}_L}{2} \end{cases} . \tag{36}$$

Hence the value of the Asian option at any time t will lie between those two values. This can be seen in figures 10-12, where the minimum and maximum Black-Scholes values (solid curves) give a good bound to the Asian option (dashed curve) for each barrier problem.
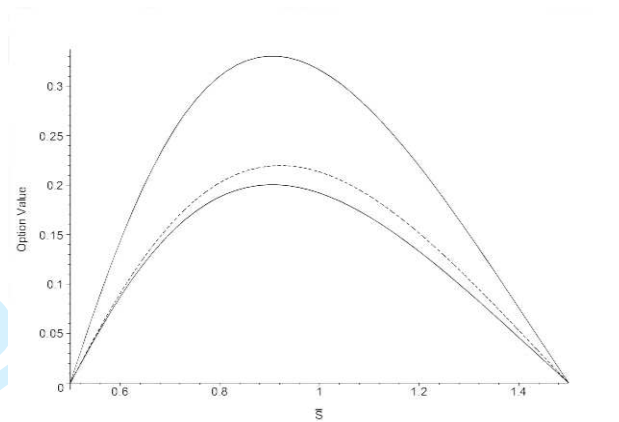
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Figure 10: Plot of the double barrier Asian option, (27), (dash line) and the minimum and maximum values (bottom and top solid lines respectively) of the Black-Scholes against $\bar{S}$ at time $t = 0$.
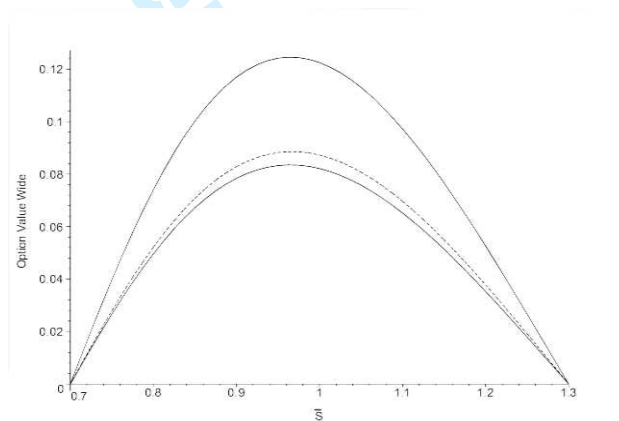
Figure 11: Plot of the double widened barrier Asian option, (29), (dash line) and the minimum and maximum values (bottom and top solid lines respectively) of the Black-Scholes against $\bar{S}$ at time $t = 0$.

4.1.1 Bounds for other payoffs

We can find bounds to the Asian option with payoff as mentioned in section (3.3). This can be done by taking three options with payoffs $V_1 = \frac{\partial}{\partial \alpha_1}(\frac{1}{T} \exp(\alpha_1 \bar{I})) = \exp(\alpha_1 \bar{I}) \frac{\bar{I}}{T}$, $V_2 = \exp(\alpha_1 \bar{I}) \bar{S}_L$ and $V_3 = \exp(\alpha_1 \bar{I}) \bar{S}_U$. T is the expiry date of 1 year and α_1 is a constant parameter taken to have

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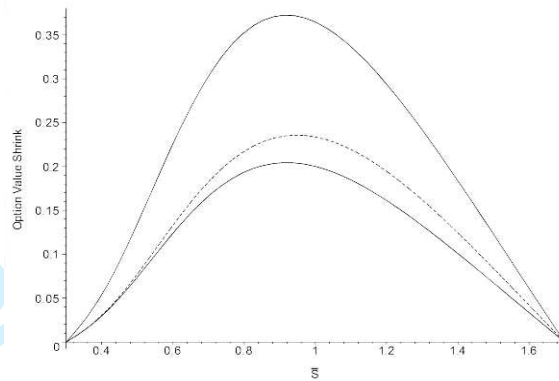
Figure 12: Plot of the double shrunken barrier Asian option, (31), (dash line) and the minimum and maximum values (bottom and top solid lines respectively) of the Black-Scholes against $\bar{S}$ at time $t = 0$.
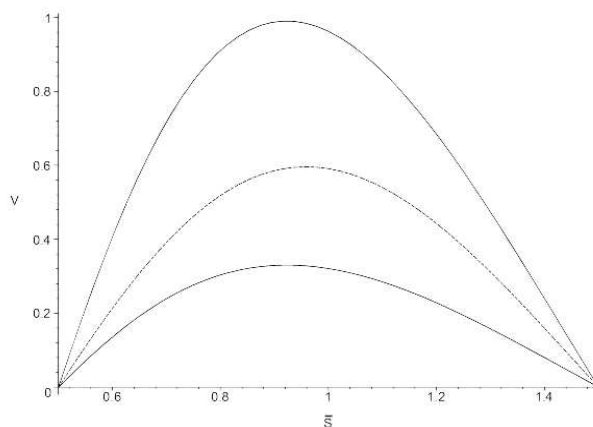

Figure 13: Bounds on the Asian option at time $t = 0$ (V_1, dash curve, V_2 bottom solid curve, V_3 top solid curve).

value equal to 0.5. We expect the value of the option V_1 to lie between the values of the options V_2 and V_3. The graphs that correspond to these options at $t = 0$ are given in figure 13.

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A Asian options with transaction costs

We consider the effect of transaction costs for the case of strongly path-dependent options and more precisely for Asian options. We take the case where a similarity variable is introduced in the problem of Geman & Yor (1994). With a model of continuous transaction costs $k_1 = k\sqrt{\frac{2}{\pi \delta t}}$ proportional to the asset value S, the partial differential equation valid for any path-dependent option is of the form

$$\frac{\partial V}{\partial t} + S\,\frac{\partial V}{\partial I} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - k_1 \sigma S^2 \left|\frac{\partial^2 V}{\partial S^2}\right| + rS\,\frac{\partial V}{\partial S} - rV = 0\,, \qquad (37)$$

where I is the running sum.
We set $V(S,I,t) = \exp\left[-r(T-t)\right] V_2(S,I,t)$ in the latter PDE,

$$\frac{\partial V_2}{\partial t} + S\,\frac{\partial V_2}{\partial I} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_2}{\partial S^2} - k_1 \sigma S^2 \left|\frac{\partial^2 V_2}{\partial S^2}\right| + rS\,\frac{\partial V_2}{\partial S} + rV_2 = 0\,. \qquad (38)$$

We then change the variable by introducing the similarity variable

$$U = \frac{1}{S}\left(K_1 - I\right), \quad \text{where} \quad K_1 = KT\,, \qquad (39)$$

K being the strike price and T the maturity date.
Hence equation (38) becomes

$$\frac{\partial W}{\partial t_1} = \frac{1}{2}\,\sigma^2 U^2 \frac{\partial^2 W}{\partial U^2} - (1+rU)\frac{\partial W}{\partial U} - k_1 \sigma \left|U^2 \frac{\partial^2 W}{\partial U^2}\right| + rW\,, \qquad (40)$$

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where $V_2 = \frac{S}{T} W(U,t)$, $t_1 = T - t$ and with payoff for a fixed-strike Asian call option

$$W(U,0) = \max(-U,0)\,. \tag{41}$$

We examine the sign the Greek gamma ($\Gamma = \frac{\partial^2 W}{\partial U^2}$) takes by returning to the PDE (40) without the effect of transaction costs and see whether we can draw any conclusion for the sign of the modulus. We have the equation

$$\frac{\partial W}{\partial t_1} = \frac{1}{2}\,\sigma^2 U^2 \frac{\partial^2 W}{\partial U^2} - (1 + rU)\frac{\partial W}{\partial U} + rW\,.$$

Differentiating the above equation with respect to U gives

$$\frac{\partial \Delta}{\partial t_1} = \frac{1}{2}\,\sigma^2 U^2 \frac{\partial^2 \Delta}{\partial U^2} + [(\sigma^2 - r)U - 1]\frac{\partial \Delta}{\partial U}\,. \tag{42}$$

When $t_1 = 0$,

$$\Delta = \frac{\partial W}{\partial U} = \left\{ \begin{array}{ll} -1 & \text{, for } U < 0 \\ 0 & \text{, for } U > 0 \end{array} \right.\,.$$

We differentiate again to get a partial differential equation for Γ,

$$\frac{\partial \Gamma}{\partial t_1} = \frac{1}{2}\,\sigma^2 U^2 \frac{\partial^2 \Gamma}{\partial U^2} + [(2\sigma^2 - r)U - 1]\frac{\partial \Gamma}{\partial U} + (\sigma^2 - r)\Gamma\,, \tag{43}$$

and at time $t_1 = 0$, $\Gamma = \frac{\partial^2 W}{\partial U^2} = 0$ everywhere apart from the point $U = 0$ where Γ is positive but undefined.
We set $\Gamma = \exp\left((\sigma^2 - r)t_1\right)\Gamma_1$ and (43) takes the form

$$\frac{\partial \Gamma_1}{\partial t_1} = \frac{1}{2}\,\sigma^2 U^2 \frac{\partial^2 \Gamma_1}{\partial U^2} + [(2\sigma^2 - r)U - 1]\frac{\partial \Gamma_1}{\partial U}\,.$$

At this stage we still cannot determine the sign of Γ so we multiply the equation by a function of μ which we will find later,

$$\begin{aligned} \mu\frac{\partial \Gamma_1}{\partial t_1} &= \frac{1}{2}\,\sigma^2 U^2 \mu\frac{\partial^2 \Gamma_1}{\partial U^2} + [(2\sigma^2 - r)U - 1]\mu\frac{\partial \Gamma_1}{\partial U} \\ &= \frac{\partial}{\partial U}\left[\frac{1}{2}\,\sigma^2 U^2 \mu\,\frac{\partial \Gamma_1}{\partial U}\right]\,. \end{aligned}$$

Provided

$$\sigma^2 U \mu + \frac{1}{2}\,\sigma^2 U^2 \frac{\partial \mu}{\partial U} = \mu[(2\sigma^2 - r)U - 1]\,,$$

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to hold, which gives

$$\mu = A\,|U|^{\frac{2(\sigma^2 - r)}{\sigma^2}} \exp\left(\frac{2}{\sigma^2 U}\right). \tag{44}$$

Substituting into the equation we have

$$|U|^{\frac{2(\sigma^2 - r)}{\sigma^2}} \exp\left(\frac{2}{\sigma^2 U}\right)\frac{\partial \Gamma_1}{\partial t_1} = \frac{\partial}{\partial U}\left[\frac{1}{2}\,\sigma^2 |U|^{\frac{2(2\sigma^2 - r)}{\sigma^2}} \exp\left(\frac{2}{\sigma^2 U}\right)\frac{\partial \Gamma_1}{\partial U}\right]. \tag{45}$$

The above equation is a non-linear diffusion equation with positive coefficients. Therefore as a consequence of (45) the gamma for the call option, $\Gamma = \frac{\partial^2 W}{\partial U^2}$, will always be positive. So we can proceed with the valuation of the Asian option with the effect of transaction costs, by removing the modulus sign. Hence equation (40) becomes

$$\frac{\partial W}{\partial t_1} = \left(\frac{1}{2}\,\sigma^2 - k_1 \sigma\right) U^2 \frac{\partial^2 W}{\partial U^2} - (1 + rU)\frac{\partial W}{\partial U} + rW\,, \tag{46}$$

with payoff for a fixed-strike Asian call option

$$W(U,0) = \max(-U, 0)\,, \tag{47}$$

and

$$W(U,0) = \max(U, 0)\,, \tag{48}$$

for a fixed-strike Asian put option.
For the case of the similarity variable $R = \frac{I}{S}$ the governing PDE with transaction costs is of the form

$$\frac{\partial H}{\partial t} + \frac{1}{2}\,\sigma^2 R^2 \frac{\partial^2 H}{\partial R^2} + (1 - rR)\frac{\partial H}{\partial R} - k_1 \sigma \left|R^2 \frac{\partial^2 H}{\partial R^2}\right| + rH = 0\,, \tag{49}$$

with payoff for a floating-strike Asian call option

$$H(R,T) = \max\left(1 - \frac{R}{T}, 0\right). \tag{50}$$

Applying the above analysis for this problem leads to the diffusion equation

$$|R|^{\frac{2(\sigma^2 - r)}{\sigma^2}} \exp\left(-\frac{2}{\sigma^2 R}\right)\frac{\partial \Gamma_1}{\partial t_1} = \frac{\partial}{\partial R}\left[\frac{1}{2}\,\sigma^2 |R|^{\frac{2(2\sigma^2 - r)}{\sigma^2}} \exp\left(-\frac{2}{\sigma^2 R}\right)\frac{\partial \Gamma_1}{\partial R}\right]. \tag{51}$$

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The delta of the payoff behaves like

$$\Delta = \frac{\partial H}{\partial R} = \left\{ \begin{array}{ll} -\frac{1}{T} & \text{, for } R < T \\ 0 & \text{, for } R > T \end{array} \right. ,$$

and the gamma of the payoff is zero everywhere apart from the point $R = T$ when it is infinite.

Following the above procedure one can determine the sign of the modulus of the equation for options where the gamma of the payoff is single signed. For the case where the sign of the gamma changes this method cannot be efficient and the problem needs to be solved numerically.

B The WKB Solution

Using separation of variables in equation (8),

$$\frac{\partial W}{\partial t_2} = \frac{\partial^2 W}{\partial \tilde{x}^2} + \alpha_3 \exp\left(\tilde{x}\right) W , \tag{52}$$

we end up with a system of ordinary differential equations, one time dependent and another one space dependent,

$$T' + k^2 T = 0 , \tag{53}$$
$$X''(\tilde{x}) + (\alpha_3 \exp\left(\tilde{x}\right) + k^2)X(\tilde{x}) = 0 , \tag{54}$$

where k^2 is the separation variable.

To apply the WKB method we write the above ODE, (54), in the form

$$\epsilon^2 y'' = Q(x)y , \quad Q(x) \neq 0 , \tag{55}$$

where $\epsilon = 1/k$ is small and tends to zero, $Q(\tilde{x}) = -(1 + \frac{\alpha_3}{k^2} \exp\left(\tilde{x}\right))$ and (55) is to be solved with boundary conditions

$$X(0, t_2) = X(\bar{x}_U - \bar{x}_L, t_2) = 0 .$$

Equation (55) is the Schrödinger equation whose approximate solution for ϵ small can be found using the WKB method; that is

$$y(x) \sim$$
$$C_1 Q(x)^{-1/4} \exp\left(\frac{1}{\epsilon} \int_a^x \sqrt{Q(t)} \, dt\right) + C_2 Q(x)^{-1/4} \exp\left(-\frac{1}{\epsilon} \int_a^x \sqrt{Q(t)} \, dt\right) ,$$

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as $\epsilon \to 0$, where C_1 and C_2 are constants to be determined from the initial and boundary conditions and a is an arbitrary but fixed integration point, e.g. Bender & Orzsag (1978). The condition $y(\tilde{x}) = 0$ at $\tilde{x} = 0$ gives C_1 proportional to C_2 and eigenvalues of k (k_n) (with associated eigenfunction y_n) are selected to satisfy the condition at $\tilde{x} = \tilde{x}_U - \tilde{x}_L$. The remaining proportional constant is to be determined from the resulting series shown below and the initial condition at $t_2 = 0$. Therefore the general solution of equation (52) will take the form

$$
\begin{aligned}
W(\tilde{x}, t_2) &= \sum_{n=1}^{\infty} A_n \exp\left(-k_n^2 t_2\right)(\alpha_3 \exp\left(\tilde{x}\right) + k_n^2)^{-1/4} \\
&\quad \sin\left(\int_0^{\tilde{x}} \sqrt{\alpha_3 \exp\left(\tau\right) + k_n^2}\, d\tau\right),
\end{aligned} \tag{56}
$$

satisfying the boundary condition at $\tilde{x} = 0$.

The exact solution of the equation can be written in terms of a conjugate pair of Bessel functions

$$
W(\tilde{x}, t_2) = \sum_{n=1}^{\infty} A_n \exp\left(-k_n^2 t_2\right) \tag{57}
$$

$$
\left\{ C_1 \Gamma(1 - 2ik_n) J_{-2ik_n}(2\sqrt{\alpha_3 \exp\left(\tilde{x}\right)}) + C_2 \Gamma(1 + 2ik_n) J_{2ik_n}(2\sqrt{\alpha_3 \exp\left(\tilde{x}\right)}) \right\}.
$$

C   Other payoffs

Differentiating equation (35) with respect to α_1 we get a complicated expression of the form

$$
V^{(2)}(\bar{S}, \bar{I}, t) = \frac{dV}{d\alpha} = \exp\left(\alpha_1 \bar{I}\right) \exp\left(-\left(\frac{r^2}{2\sigma^2} + \frac{\sigma^2}{8} + \frac{r}{2}\right)(T-t)\right) \bar{S}^{\left(\frac{1}{2} - \frac{r}{\sigma^2}\right)}
$$

$$
\left[\frac{\bar{I}}{T} \sum_{n=1}^{\infty} A_n \exp\left(-\frac{\sigma^2}{2} k_n^2 (T-t)\right) \left(\frac{2\alpha_1}{\sigma^2} \bar{S} + k_n^2\right)^{-1/4}\right. \tag{58}
$$

$$
\sin\left(\int_1^{\bar{S}/\bar{S}_L} \sqrt{\frac{2\alpha_1}{\sigma^2} \bar{S}_L v + k_n^2} \frac{dv}{v}\right) + \frac{1}{T} \sum_{n=1}^{\infty} \frac{dA_n}{d\alpha_1} \exp\left(-\frac{\sigma^2}{2} k_n^2 (T-t)\right)
$$

$$
\left(\frac{2\alpha_1}{\sigma^2} \bar{S} + k_n^2\right)^{-1/4} \sin\left(\int_1^{\bar{S}/\bar{S}_L} \sqrt{\frac{2\alpha_1}{\sigma^2} \bar{S}_L v + k_n^2} \frac{dv}{v}\right)
$$

$$
- \frac{\sigma^2 (T-t)}{T} \sum_{n=1}^{\infty} A_n k_n \frac{dk_n}{d\alpha_1} \exp\left(-\frac{\sigma^2}{2} k_n^2 (T-t)\right) \left(\frac{2\alpha_1}{\sigma^2} \bar{S} + k_n^2\right)^{-1/4}
$$

$$
\sin\left(\int_1^{\bar{S}/\bar{S}_L} \sqrt{\frac{2\alpha_1}{\sigma^2} \bar{S}_L v + k_n^2} \frac{dv}{v}\right) - \frac{1}{2T} \sum_{n=1}^{\infty} A_n \exp\left(-\frac{\sigma^2}{2} k_n^2 (T-t)\right)
$$

$$
\left(\frac{\bar{S}}{\sigma^2} + k_n \frac{dk_n}{d\alpha}\right) \left(\frac{2\alpha_1}{\sigma^2} \bar{S} + k_n^2\right)^{-5/4} \sin\left(\int_1^{\bar{S}/\bar{S}_L} \sqrt{\frac{2\alpha_1}{\sigma^2} \bar{S}_L v + k_n^2} \frac{dv}{v}\right)
$$

$$
+ \frac{1}{T} \sum_{n=1}^{\infty} A_n \exp\left(-\frac{\sigma^2}{2} k_n^2 (T-t)\right) \left(\frac{2\alpha_1}{\sigma^2} \bar{S} + k_n^2\right)^{-1/4}
$$

$$
\cos\left(\int_1^{\bar{S}/\bar{S}_L} \sqrt{\frac{2\alpha_1}{\sigma^2} \bar{S}_L v + k_n^2} \frac{dv}{v}\right)
$$

$$
\left.\int_1^{\bar{S}/\bar{S}_L} \left(\frac{\bar{S}_L}{\sigma^2} v + k_n \frac{dk_n}{d\alpha}\right) \left(\frac{2\alpha_1}{\sigma^2} \bar{S}_L v + k_n^2\right)^{-1/2} \frac{dv}{v}\right],
$$

where the eigenvalues k_n, are given by the formula

$$
\int_1^{\bar{S}_U/\bar{S}_L} \sqrt{\frac{2\alpha_1}{\sigma^2} \bar{S}_L v + k_n^2} \frac{dv}{v} = n\pi,
$$

the derivative $\frac{dk_n}{d\alpha_1}$, for each k_n, by differentiating the latter integral equation with respect to α_1,

$$
k_n \frac{dk_n}{d\alpha_1} = -\frac{\bar{S}_L}{\sigma^2} \frac{\int_1^{\bar{S}_U/\bar{S}_L} \left(\frac{2\alpha_1}{\sigma^2} \bar{S}_L v + k_n^2\right)^{-1/2} dv}{\int_1^{\bar{S}_U/\bar{S}_L} \left(\frac{2\alpha_1}{\sigma^2} \bar{S}_L v + k_n^2\right)^{-1/2} \frac{dv}{v}}, \tag{59}
$$

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and the derivative $\frac{dA_n}{d\alpha_1}$ by differentiating the coefficients A_n with respect to α_1, with $W(\bar{S}, T) = F(\bar{S}) = 1$, will be equal to

$$
\frac{dA_n}{d\alpha_1} = \left[\left(\int_{\bar{S}_L}^{\bar{S}_U} [W_n(\bar{S})]^2 \, d\bar{S} \right) \int_{\bar{S}_L}^{\bar{S}_U} \frac{d}{d\alpha_1} [W_n(\bar{S})] \, d\bar{S} - \tag{60}
$$
$$
\left(\int_{\bar{S}_L}^{\bar{S}_U} W_n(\bar{S}) \, d\bar{S} \right) \int_{\bar{S}_L}^{\bar{S}_U} \frac{d}{d\alpha_1} [W_n(\bar{S})]^2 \, d\bar{S} \right] / \left(\int_{\bar{S}_L}^{\bar{S}_U} [W_n(\bar{S})]^2 \, d\bar{S} \right)^2 .
$$

The expression, (58), is proportional to the average $\bar{I}$ giving a term $\bar{I} \exp{(\alpha_1 \bar{I})} g(\bar{S})$ plus another term of the form $\exp{(\alpha_1 \bar{I})} g(\bar{S})$. The solution to the problem of double barrier Asian options corresponds to the payoff (18).

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