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Henry-Labordere, Pierre

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SOLVABLE LOCAL AND STOCHASTIC VOLATILITY MODELS: SUPERSYMMETRIC METHODS IN OPTION PRICING

PIERRE HENRY-LABORDÈRE

ABSTRACT. In this paper we provide an extensive classification of one and two dimensional diffusion processes which admit an exact solution to the Kolmogorov (and hence Black-Scholes) equation (in terms of hypergeometric functions). By identifying the one-dimensional solvable processes with the class of *integrable superpotentials* introduced recently in supersymmetric quantum mechanics, we obtain new analytical solutions. In particular, by applying *supersymmetric transformations* on a known solvable diffusion process (such as the Natanzon process for which the solution is given by a hypergeometric function), we obtain a hierarchy of new solutions. These solutions are given by a sum of hypergeometric functions, generalizing the results obtained in the paper "Black-Scholes Goes Hypergeometric" [1]. For two-dimensional processes, more precisely stochastic volatility models, the classification is achieved for a specific class called gauge-free models including the Heston model, the 3/2-model and the geometric Brownian model. We then present a new exact stochastic volatility model belonging to this class.

1. INTRODUCTION

For most mathematical models of asset dynamics, an exact solution for the corresponding Kolmogorov & Black-Scholes equation is usually not available: there are, however, a few notable exceptions. The known solutions for local volatility models are the constant elastic of variance (CEV) [6] including the classical log-normal Black-Scholes process. For the instantaneous short rate models, there are the CIR process [7] (Bessel process) and the Vasicek-Hull-White process [9] (Ornstein-Uhlenbeck process). For stochastic volatility models, the known exact solutions are the Heston model [8], the 3/2-model and the geometric Brownian model [11]. These analytical solutions can be used for calibrating a model quickly and efficiently or can serve as a benchmark for testing the implementation of more realistic models requiring intensive numerical computation (Monte-Carlo, partial differential equation). For example, the existence of a closed-form solution for the price of a vanilla option in the Heston model allows us to quickly calibrate the model to the implied volatilities observed on the market. The calibrated model can then be used to value path-dependent exotic options using, for example, a Monte-Carlo methodology.

In this paper, we show how to obtain new analytic solutions to the Kolmogorov & Black-Scholes equation, which we refer to as KBS throughout the rest of the paper, for 1d & 2d diffusion processes. In order to get to our classification, we first present a general reduction method to simplify the multi-dimensional KBS equation. Rewriting the KBS equation as a heat kernel equation on a Riemannian manifold endowed with an Abelian connection, we show that this covariant equation can be simplified using both the group of diffeomorphisms (i.e. change of variables) and the group of Abelian gauge transformations. In particular for the models admitting a *flat Abelian connection*, there always exists a gauge transformation that eliminates the Abelian connection of the diffusion operator.

1

2   PIERRE HENRY-LABORDÈRE

In the second part we apply the reduction method, previously presented, to a one-dimensional, time-homogeneous diffusion process. Modulo a change of variable, the metric becomes flat and the Abelian connection is an exact one-form for which a gauge transformation can always be applied. Using these two transformations, the resulting KBS equation becomes a *Euclidean Schrödinger equation* with a scalar potential. Extensive work has already been done to classify the set of scalar potentials which admit an exact solution. In particular, a classification of one-dimensional time-homogeneous solvable diffusion processes for which the solution to the KBS equation can be written as a hypergeometric function has been achieved in [1, 2, 3] using the well known Natanzon classification [15].

Using a supersymmetric formulation of the Schrödinger equation which consists in doubling the BKS equation with another equation, we show how to generate a hierarchy of new solvable diffusion processes. In this context, the local volatility function is identified with a superpotential. Applying supersymmetric transformations on the Natanzon potential (which is the most general potential for which the Schrödinger equation can be reduced to either a hypergeometric or a confluent equation), we obtain a new class of solvable one-dimensional diffusion processes which are characterized by six parameters.

In the last part, we pursue this classification for stochastic volatility models which admit a flat Abelian connection: we refer to these as gauge-free models. Surprisingly, this class includes all the well known exact stochastic volatility models (i.e. the Heston model, the 3/2-model and the geometric Brownian model). For these gauge-free models, we reduce the two-dimensional KBS equation to a one-dimensional Euclidean Schrödinger equation with a scalar potential. Then, we present a new exact stochastic volatility model which is a combination of the Heston and 3/2-models.

2. REDUCTION METHOD

In this section, we explain how to simplify the KBS equation. This reduction method will be used in the next section to classify the solvable one and two dimensional time-homogeneous processes. This method is already well known for a one-dimensional process and is presented in [4, 12, 13] for example. However, the extension of this method to multi-dimensional diffusion processes (as two dimensional processes) requires the introduction of differential geometric objects such as a metric and an Abelian connection on a Riemannian manifold, as we will presently explain.

Let us assume that our time-homogeneous multi-dimensional diffusion model depends on n random processes x_i which can be either traded assets or hidden Markov processes (such as a stochastic volatility a or an instantaneous short rate r). Let us denote $x = (x_i)_{i=1,\cdots,n}$, with initial conditions $\alpha = (\alpha_i)_{i=1,\cdots,n}$. These variables x_i satisfy the following stochastic differential equations (SDE)

$$dx^i = b^i(x)dt + \sigma^i(x)dW_i$$
$$dW_i dW_j = \rho_{ij}(x)dt$$

with the initial condition $x_i(t = 0) = \alpha_i$. The no-arbitrage condition implies that there exists an equivalent measure $\mathbb{P}$ such that the traded assets (i.e. forwards) are (local) martingales under this measure. For $\mathbb{P}$, the drifts b_i are consequently zero for the traded assets. Note that the measure $\mathbb{P}$ is not unique as the market is not necessarily complete. Finally, the fair value of a (European) option, with payoff $f(x^i)$ at maturity T, is given by the discounted mean value of the payoff f conditional on the filtration $\mathcal{F}_t$ generated by the Brownian motions $\{W^i_{s \leq t}\}$

$$\mathcal{C}(\alpha, t, T) = \mathbb{E}^{\mathbb{P}}[e^{-\int_t^T r_s ds} f | \mathcal{F}_t]$$

with r_s the instantaneous short rate. This mean-value depends on the probability density $p(x, \tau | \alpha)$ which satisfies the backward Kolmogorov equation ($\tau = T - t$, $\partial_i = \frac{\partial}{\partial \alpha_i}$)

$$(1) \qquad \frac{\partial p}{\partial \tau} = b^i \partial_i p + \frac{1}{2} \rho_{ij} \sigma^i \sigma^j \partial_{ij} p$$

with initial condition $p(x, \tau = 0 | \alpha) = \delta(x - \alpha)$. In this equation we have used the Einstein convention meaning that two repeated indices are summed.

Using the Feymann-Kac theorem, one can show that the fair value $\mathcal{C}$ for the option satisfies the Black-Scholes equation ($\partial_i = \frac{\partial}{\partial \alpha_i}$)

$$(2) \qquad \frac{\partial \mathcal{C}}{\partial \tau} = b^i \partial_i \mathcal{C} + g^{ij} \partial_{ij} \mathcal{C} - r(\alpha) \mathcal{C}$$

with initial condition $\mathcal{C}(\tau = 0, \alpha) = f(\alpha)$.

In the following, the partial differential equations (PDE) (1,2) will be interpreted as a heat kernel on a general smooth n-dimensional manifold M (without a boundary) endowed with a metric g_{ij}. The inverse of the metric g^{ij} is defined by

$$g^{ij} = \frac{1}{2} \rho_{ij} \sigma^i \sigma^j$$

and the metric (ρ^{ij} inverse of ρ_{ij})

$$g_{ij} = 2 \frac{\rho^{ij}}{\sigma^i \sigma^j}$$

The differential operator

$$(3) \qquad D = b^i \partial_i + g^{ij} \partial_{ij}$$

which appears in (1) is a second order elliptic operator of Laplace type. We can then show that there is a unique connection ∇ on $\mathcal{L}$, a line bundle over M, and a unique smooth section Q of $\mathcal{L}$ such that

$$(4) \qquad \begin{aligned} D &\equiv g^{ij} \nabla_i \nabla_j + Q \\ &= g^{-\frac{1}{2}} (\partial_i + \mathcal{A}_i) g^{\frac{1}{2}} g^{ij} (\partial_j + \mathcal{A}_j) + Q \end{aligned}$$

with $g = det[g_{ij}]$. We may express the connection $\mathcal{A}^i$ and Q as a function of the drift b_i and the metric g_{ij} by identifying in (4) the terms ∂_i and ∂_{ij} with those in (3). We find

$$(5) \qquad \mathcal{A}^i = \frac{1}{2}(b^i - g^{-\frac{1}{2}} \partial_j (g^{1/2} g^{ij}))$$

$$(6) \qquad Q = g^{ij} (\mathcal{A}_i \mathcal{A}_j - b_j \mathcal{A}_i - \partial_j \mathcal{A}_i)$$

Note that the Latin indices $i, j \cdots$ can be lowered or raised using the metric g_{ij} or its inverse g^{ij}. For example $\mathcal{A}_i = g_{ij} \mathcal{A}^j = \frac{1}{2}(g_{ik} b^k - \frac{1}{2} \partial_i \ln(g) - g_{ip} \partial_k g^{pk})$. Using this connection, (1) can be rewritten in the covariant way, i.e.

$$(7) \qquad \frac{\partial}{\partial \tau} p(x, \tau | \alpha) = D p(x, \tau | \alpha)$$

If we take $\mathcal{A}_i = 0$, $Q = 0$ then D becomes the Laplace-Beltrami operator (or Laplacian) $\Delta = g^{-\frac{1}{2}} \partial_i g^{\frac{1}{2}} g^{ij} \partial_j$. For this configuration, (7) will be called the Laplacian heat kernel. Similarly, the Black-Scholes equation (2) can be rewritten

$$\frac{\partial}{\partial \tau} \mathcal{C}(\alpha, \tau) = (D - r) \mathcal{C}(\alpha, \tau)$$

The heat kernel equation can now be simplified by applying the actions of the following groups:

4   PIERRE HENRY-LABORDÈRE

$\triangleright$ The group of diffeomorphisms $Diff(\mathcal{M})$ which acts on the metric g_{ij} and the connection $\mathcal{A}_i$ by

$$\begin{aligned} f^* g_{ij} &= g_{pk} \partial_i f^p(x) \partial_j f^k(x) \\ f^* \mathcal{A}_i &= \mathcal{A}_p \partial_i f^p(x) \ , \ f \in Diff(\mathcal{M}) \end{aligned}$$

$\triangleright$ The group of gauge transformations $\mathcal{G}$ which acts on the conditional probability (and the fair value $\mathcal{C}$) by

$$\begin{aligned} p'(x,\tau|\alpha) &= e^{\chi(x,\tau)-\chi(x=\alpha,\tau=0)} p(x,\tau|\alpha) \\ \mathcal{C}'(\alpha,\tau) &= e^{\chi(\alpha,\tau)} \mathcal{C}(\alpha,\tau) \end{aligned}$$

Then p' ($\mathcal{C}'$) satisfies the same equation as p ($\mathcal{C}$) (7) only with

$$\begin{aligned} \mathcal{A}'_i &\equiv \mathcal{A}_i - \partial_i \chi \\ Q' &\equiv Q + \partial_\tau \chi \end{aligned}$$

The constant phase $e^{\chi(x=\alpha,\tau=0)}$ has been added so that p and p' satisfy the same boundary condition at $\tau = 0$. The above transformation is called a *gauge transformation*. If the connection $\mathcal{A}$ is an exact form (meaning that there exists a smooth function Λ such that $\mathcal{A}_i = \partial_i \Lambda$), then by applying a gauge transformation, we can eliminate the connection so that the heat kernel equation for p' (or $\mathcal{C}'$) has a connection equal to zero. It can be shown that for a simply-connected manifold, the statement "$\mathcal{A}$ is exact" is equivalent to $\mathcal{F} = 0$, where $\mathcal{F}$ is the 2-form curvature given in a specific coordinate system by

$$\mathcal{F}_{ij} = \partial_i \mathcal{A}_j - \partial_j \mathcal{A}_i$$

In the following, we will restrict our classification to those processes for which $\mathcal{F} = 0$, meaning there exists a gauge transformation such that the transformed connection vanishes. The operator D reduces in this case to the symmetric operator $D = \Delta + Q$ for which we can use an eigenvector expansion.

Spectral decomposition

This spectral decomposition is valid only if the symmetric operator D is an (unbounded) self-adjoint operator or admits a self-adjoint extension. This will depend strongly on the boundary conditions. After proving that D is a self-adjoint operator or admits a self-adjoint extension, the conditional probability (or the fair value) can then be expanded over a complete basis of orthonormal eigenvectors $\phi_n(x)$

$$p(x,\tau|\alpha) = \sum_n e^{-E_n \tau} \phi_n(x) \phi_n(\alpha)$$

with $D\phi_n(x) = E_n \phi_n(x)$. The discrete summation over n can also include a continuous summation according to a specific measure $\mu(E)$ if the spectrum contains a continuous part (we speak about a spectral resolution of identity).

3. 1D TIME-HOMOGENEOUS DIFFUSION MODELS

In the next section, we apply the general reduction method, presented previously, to the one-dimensional KBS equation. Similar reduction to a Schrödinger equation with a scalar potential (without any references to differential geometry) can be found in [4, 12, 13]. However, we rederive this classical result using our general reduction method as we will use it when we will discuss the classification of stochastic volatility models. We then find the supersymmetric partner of this Schrödinger equation and show how to generate new exact solutions (for Vanilla options).

3.1. **Reduction method.** Consider a one-dimensional, time-homogeneous diffusion process with drift [1]

$$df = \mu(f)dt + \sigma(f)dW$$

This process has been used as the basis for various mathematical models in finance. If f is a traded asset (i.e. a forward), the drift vanishes in the forward measure and we have a local volatility model where we assume that the volatility is only a function of f. The one-dimensional process is not necessarily driftless as the random variable is not a traded asset as it is the case for an instantaneous short rate model, or a model of stochastic volatility. In our framework, this process corresponds to a (one-dimensional) real curve endowed with the metric $g_{ff} = \frac{2}{\sigma(f)^2}$. For the new coordinate

$$s(f) = \sqrt{2} \int_{f_0}^{f} \frac{df'}{\sigma(f')}$$

the metric is flat: $g_{ss} = 1$. The Laplace-Beltrami operator therefore becomes $\triangle = \partial_s^2$. Using the definitions (5), (6), we find that the Abelian connection $\mathcal{A}_f$ and the function Q are given by

$$\mathcal{A}_f = -\frac{1}{2}\partial_f \ln(\sigma(f)) + \frac{\mu(f)}{\sigma^2(f)}$$

$$Q = \frac{1}{4}(\sigma(f)\sigma''(f) - \frac{1}{2}\sigma'(f)^2) - \frac{\mu'(f)}{2} + \frac{\mu(f)\sigma'(f)}{\sigma(f)} - \frac{\mu(f)}{2\sigma^2(f)}$$

In this case, by applying a gauge transformation on the conditional probability p, $P = \frac{\sigma(f_0)}{\sqrt{2}}e^{\Lambda}p$ with

$$\Lambda = -\frac{1}{2}\ln(\frac{\sigma(f)}{\sigma(f_0)}) + \int_{f_0}^{f} \frac{\mu(f')}{\sigma^2(f')}df'$$

then the connection vanishes and P satisfies a heat kernel with a scalar potential $Q(s)$ (in the s flat coordinate)

$$(8) \qquad \partial_\tau P(s,\tau) = (\partial_s^2 + Q(s))P(s,\tau)$$

The solution P has been scaled by the (constant) factor $\frac{\sigma(f_0)}{\sqrt{2}}$ in order to have the initial condition $lim_{\tau \to 0}P(s,\tau) = \delta(s)$. Moreover, Q is given in the s coordinate by

$$(9) \qquad Q = \frac{1}{2}(\ln(\sigma))''(s) - \frac{1}{4}((\ln(\sigma))'(s))^2 - \frac{\mu'(s)}{\sqrt{2}\sigma(s)} + \frac{\sqrt{2}\mu(s)\sigma'(s)}{\sigma(s)^2} - \frac{\mu^2(s)}{2\sigma^2(s)}$$

where the prime $'$ indicates a derivative according to s.

Example 3.1.1 (quadratic volatility process). Let us assume that f satisfies a driftless process (i.e. $\mu(f) = 0$). The Black-Scholes equation reduces to the heat kernel on $\mathbb{R}$ if $Q(s) = constant$ (i.e. $Q(s)$ is zero modulo a time-dependent gauge transformation) which is equivalent to $\sigma(f) = \alpha f^2 + \beta f + \gamma$ (i.e. the quadratic volatility model, also called the hyperbolic model [12]).

Example 3.1.2 (CEV process). For the CEV process $df = f^{\beta}dW_t$ $\mu(f) = 0$, the potential is $Q(s) = \frac{\beta(\beta-2)}{4(1-\beta)^2 s^2}$ for $\beta \neq 1$ and $Q(s) = -\frac{1}{8}$ for $\beta = 1$.

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[1]The time-dependent process $df = \mu(f)A^2(t)dt + \sigma(f)A(t)dW$ is equivalent to this process under the change of local time $t' = \int^t A^2(s)ds$

If the Hamiltonian $D = (-\partial_s^2 - Q)$ is self-adjoint or admits self-adjoint extensions, the spectral decomposition can be used and the conditional probability can be decomposed using a complete basis of orthonormal eigenvectors $\phi_n(s)$:

$$(10) \qquad P(s,\tau) = \sum_n \phi_n(s)\phi_n^\dagger(s_0)e^{-E_n\tau}$$

with $\phi_n(s)$ satisfying

$$(11) \qquad D\phi_n(s) = E_n\phi_n(s)$$

In the next section, we explain how to obtain the fair value of an European call option (in the case of constant interest rate) from this spectral decomposition (10).

3.2. **Vanilla option with constant interest rate.** We have that the forward f is a local martingale (i.e. $\mu(f) = 0$) in the forward measure. The value at t of a European option (with strike K and expiry date T) is then given by ($\tau = T - t$)

$$\mathcal{C}(f_t, K, \tau) = e^{-r\tau}\int_K^\infty (f - K)p(f, \tau|f_t)df$$

Doing an integration by parts, or equivalently, applying the Ito-Tanaka-Meyer formula on the payoff $(S_t - K)^+$ [3] we can show that the vanilla option $\mathcal{C}$ can be rewritten as

$$(12) \qquad \mathcal{C}(f_t, K, \tau) = e^{-r\tau}(f_t - K)^+ + e^{-r\tau}\frac{\sigma^2(K)}{2}\int_0^\tau dt'p(K, t'|f_t)$$

Using the relation between the conditional probability $p(f, t'|f_t)$ and its gauge-transform $P(s(f), t'|s_t)$ in (12), we obtain

$$(13) \qquad \mathcal{C}(f_t, K, \tau) = e^{-r\tau}((f_t - K)^+ + \frac{\sqrt{\sigma(K)\sigma(f_t)}}{2}\int_0^\tau P(s(K), t'|s_t)dt')$$

Plugging the expression for $P(s, t'|s_t)$ (10) into (13) and doing the integration over the time t, we obtain [3]

$$\mathcal{C}(f_t, K, \tau) = e^{-r\tau}((f_t - K)^+ + \frac{\sqrt{\sigma(K)\sigma(f_t)}}{2}\sum_n \phi_n(s(K))\phi_n(s_t)\frac{(1 - e^{-E_n\tau})}{E_n})$$

A specific local volatility model will give an exact solution for a vanilla option if we can find the eigenvalues and eigenvectors for the corresponding Euclidean Schrödinger equation with a scalar potential. As examples of solvable potentials, one can cite the harmonic oscillator, Coulomb, Morse, Pöschl-Teller I&II, Eckart and Manning-Rosen potentials. The classification of solvable scalar potentials was initiated by Natanzon [15]. This work provides the most general potential for which the Schrödinger equation can be reduced to either a hypergeometric or confluent equation. We will review in the following section the *Natanzon potential*, which depends on 6 parameters. We will later show that the Schrödinger equation can be doubled into a set of two independent Schrödinger equations with two different scalar potentials which transform into each other under a supersymmetric transformation. Moreover, if one scalar potential is solvable, the other one is. Applying this technique to the Natanzon potential, we will obtain a new class of solvable potentials corresponding to a new class of solvable diffusion processes. For these models, the solution to the KBS equation is given by a *sum* of hypergeometric functions generalizing the paper ([1]).

3.3. **Solvable (super)potentials.** In this section, we show that the KBS equation can be formulated using supersymmetric techniques. For details, the application of supersymmetric techniques to the classification of solvable potentials for the Schrödinger equation has been reviewed in [5] where a large number of references can be found. For the Kolmogorov & Fokker-Planck equations, one can consult [10].

In the following, the local volatility will be identified as a superpotential. Using this formalism, we show how to generate a hierarchy of solvable diffusion models starting from a known solvable superpotential, for example the hypergeometric or confluent hypergeometric Natanzon superpotential.

3.3.1. *Superpotential and local volatility.* Let us write the equation (11) in the following way by introducing the first-order operator A_1 and its formal adjoint $A_1^\dagger$ [2]

(14) $$E_n^{(1)}\phi_n^{(1)} = A_1^\dagger A_1 \phi_n^{(1)}$$

with $A_1 = \partial_s + W^{(1)}(s)$, $A_1^\dagger = -\partial_s + W^{(1)}(s)$. $W^{(1)}$ is called the *superpotential* which satisfies the Riccatti equation

$$Q^{(1)}(s) = \partial_s W^{(1)}(s) - W^{(1)^2}(s)$$

Surprisingly, this equation is trivially solved for our specific expression for $Q^{(1)}$ (9) (even with a drift $\mu(f)$!)

(15) $$W^{(1)}(s) = \frac{1}{2}\frac{d\ln\sigma^{(1)}(s)}{ds} - \frac{\mu^{(1)}(s)}{\sqrt{2}\sigma^{(1)}(s)}$$

In particular, for zero drift, the local volatility function is directly related to the superpotential by $\sigma(s) = e^{2\int^s W(z)dz}$.

If we have a family of solvable superpotentials $W_{solvable}^{(1)}(s)$, we can always find an analytic solution to the Kolmogorov equation for any diffusion term $\sigma^{(1)}(s)$ by adjusting the drift with the relation (15)

$$\mu^{(1)}(s) = \frac{\sigma^{(1)'}(s)}{\sqrt{2}} - \sqrt{2}\sigma^{(1)}(s)W_{solvable}^{(1)}(s)$$

Note that (14) admits a zero eigenvalue if and only if the Kolmogorov equation admits a stationary distribution. By observing that $A_1^\dagger A_1 \phi_0^{(1)} = 0$ is equivalent to $A_1 \phi_0^{(1)} = 0$, we obtain the stationary distribution

(16) $$\phi_0^{(1)}(s) = Ce^{-\int^s W^{(1)}(z)dz}$$

with C a normalization constant. Therefore, the stationary distribution will exist if the superpotential is normalisable (i.e. $\phi_0^{(1)}(s) \in L^2$).

Example 3.3.1 (Coulomb superpotential and CEV process). The CEV process corresponding to $\sigma(f) = f^\beta$ and $\mu(f) = 0$ has the *Coulomb superpotential* $W(s) = \frac{\beta}{2(1-\beta)s}$.

Next, we define the *Scholes-Black equation* by intervening the operator A_1 and $A_1^\dagger$

$$\begin{aligned}E_n^{(2)}\phi_n^{(2)}(s) &= A_1 A_1^\dagger \phi_n^{(2)}(s)\\ (17) &= (-\partial_s^2 - Q^{(2)}(s))\phi_n^{(2)}(s)\end{aligned}$$

---

[2]In order to obtain the correct adjoint operator on $\mathbb{R}^+$, we impose the absorbing boundary condition $\phi_n(s = 0) = 0$

PIERRE HENRY-LABORDÈRE

We obtain a new Schrödinger equation with the partner potential

(18) $$Q^{(2)}(s) = -\partial_s W^{(1)} - (W^{(1)})^2$$

Plugging our expression for the superpotential (15) in (18), we have

(19) $$Q^{(2)} = -\frac{1}{2}(\ln(\sigma^{(1)}))''(s) - \frac{1}{4}(\ln(\sigma^{(1)}))'(s))^2 + \frac{\mu^{(1)'}(s)}{\sqrt{2}\sigma^{(1)}(s)} - \frac{\mu^{(1)^2}(s)}{2\sigma^{(1)^2}(s)}$$

In the same way as before, H_2 admits a zero eigenvector (i.e. stationary distribution) if $\phi_0^{(2)}(s) = Ce^{\int^s W^{(1)}(z)dz}$ is normalisable.

Remark 3.3.2. In physics, the supersymmetry (SUSY) is said to be broken if at least one of the eigenvectors $\phi_0^{(1,2)}(s)$ exists. Otherwise, SUSY is said to be broken dynamically.

Now we want to show that if we can solve the equation (14) then we have automatically a solution to (17) and vice-versa. The SUSY-partner Hamiltonians $H_1 = A_1^\dagger A_1$ and $H_2 = A_1 A_1^\dagger$ obey the relations $A_1^\dagger H_2 = H_1 A_1^\dagger$ and $H_2 A_1 = A_1 H_1$. As a consequence H_1 and H_2 are isospectral. More precisely, the strictly positive eigenvalues all coincide and the corresponding eigenvectors are related by the supercharge operators A_1 and $A_1^\dagger$:

▷ If H_1 admits a zero eigenvalue (i.e. broken supersymmetry), we have the relation

$$E_n^{(2)} = E_{n+1}^{(1)} \; ; \; E_0^{(1)} = 0; \; \phi_0^{(1)}(s) = Ce^{-\int^s W^{(1)}(z)dz}$$
$$\phi_n^{(2)}(s) = (E_{n+1}^{(1)})^{-\frac{1}{2}} A_1 \phi_{n+1}^{(1)}(s)$$
$$\phi_n^{(1)}(s) = (E_n^2)^{-\frac{1}{2}} A_1^\dagger \phi_n^{(2)}(s)$$

▷ If H_1 (and H_2) doesn't admit a zero eigenvalue (i.e. unbroken supersymmetry)

$$E_n^{(2)} = E_n^{(1)}$$
$$\phi_n^{(2)}(s) = (E_n^{(1)})^{-\frac{1}{2}} A_1 \phi_n^{(1)}(s)$$
$$\phi_n^{(1)}(s) = (E_n^2)^{-\frac{1}{2}} A_1^\dagger \phi_n^{(2)}(s)$$

In the unbroken SUSY case, there are no zero modes and consequently the spectrum of H_1 and H_2 are the same. One can then obtain the solution to the Scholes-Black (resp. Black-Scholes) equation if the eigenvalues/eigenvectors of the Black-Scholes (resp. Scholes-Black) are known. We clarify this correspondence by studying a specific example: the CEV process $df = f^\beta dW$. In particular, we show that for $\beta = \frac{2}{3}$, the partner superpotential vanishes. It is therefore simpler to solve the Scholes-Black equation as Scholes-Black (rather than Black-Scholes) reduces to the heat kernel on $\mathbb{R}^+$. Applying a supersymmetric transformation on the Scholes-Black equation, we can then derive the solution to the Black-Scholes equation.

Example 3.3.3 (CEV with $\beta = 2/3$ and Bachelier process). We saw previously that the superpotential associated with the CEV process is given by

$$W^{(1)}(s) = \frac{\beta}{2s(1-\beta)}$$

with the flat coordinate $s = \frac{\sqrt{2}f^{1-\beta}}{(1-\beta)}$ and the potential (9)

$$Q^{(1)}(s) = \frac{\beta(\beta-2)}{4(1-\beta)^2 s^2}$$

from which we deduce the partner potential (19)

$$Q^{(2)}(s) = \frac{\beta(2 - 3\beta)}{4(1 - \beta)^2 s^2}$$

This partner potential corresponds to the potential of a CEV process $df = f^B dW$ with B given as a function of β by

$$\frac{B(B - 2)}{(1 - B)^2} = \frac{\beta(2 - 3\beta)}{(1 - \beta)^2}$$

Surprisingly, we observe that for $\beta = \frac{2}{3}$, $Q^{(2)}(s)$ cancels and the corresponding partner local volatility model is the Bachelier model $df = dW$ for which the heat kernel is given by the normal distribution. The eigenvectors of the supersymmetric Hamiltonian partner $H_2 = -\partial_s^2$ to H_1 are given by (with the absorbing boundary condition $\phi_n(0) = 0$)

$$\phi^{(2)}(s, E) = \frac{\sin(\sqrt{E}s)}{\sqrt{4\pi}E^{\frac{1}{4}}}$$

with a continuous spectrum $\mathbb{R}^+$. Applying the supersymmetric transformation (20), we obtain the eigenvectors for the Hamiltonian $H_1 = -\partial_s^2 + \frac{2}{s^2}$ corresponding to the CEV process with $\beta = \frac{2}{3}$

$$
\begin{aligned}
\phi^{(1)}(s, E) &= E^{-\frac{1}{2}}(-\partial_s + \frac{1}{s})\phi^{(2)}(s, E) \\
&= \frac{1}{\sqrt{4\pi}E^{\frac{3}{4}}}(-\sqrt{E}\cos(\sqrt{E}s) + \frac{\sin(\sqrt{E}s)}{s})
\end{aligned}
$$

Plugging this expression in (14), we obtain the fair value for a vanilla option

$$e^{r\tau}\mathcal{C}(f_t, K, \tau) = (f_t - K)^+ + \frac{\sqrt{Kf_t}}{2}\int_0^\infty dE \frac{(1 - e^{-E\tau})}{E}\phi^{(1)}(s, E)\phi^{(1)}(s_0, E)$$

which can be integrated and written in terms of the cumulative distribution [12]. The fact that the CEV model with $\beta = \frac{2}{3}$ depends on the normal cumulative distribution and is therefore related to the heat kernel on $\mathbb{R}^+$ has been observed empirically by [12]. Here we have seen that it corresponds to the fact that the supersymmetric partner potential vanishes for this particular value of β.

3.4. **Hierarchy of solvable diffusion processes.** In the previous section we saw that the operators A_1 and $A_1^\dagger$ can be used to factorize the Hamiltonian H_1. These operators depend on the superpotential $W^{(1)}$ which is determined once we know the first eigenvector $\phi_0^{(1)}(s)$ of H_1 (16). We have assumed that H_1 admits a zero eigenvalue. By shifting the energy E it is always possible to achieve this condition. The partner Schrödinger equation (17) can then be recast into a Schrödinger equation with a zero eigenvalue

$$H_{(2)} = A_1 A_1^\dagger - E_1^{(1)} = A_2^\dagger A_2$$

where $A_2 \equiv \partial_s + W_2(s)$ and $A_2^\dagger \equiv -\partial_s + W_2(s)$,

(20) $$W^{(2)}(s) \equiv \frac{1}{2}\frac{d \ln \sigma^{(2)}(s)}{ds} - \frac{\mu^{(2)}(s)}{\sqrt{2}\sigma^{(2)}(s)}$$

We have introduced the notation that in $E_n^{(m)}$, n denotes the energy level and (m) refers to the mth Hamiltonian H_m. By construction, this new Hamiltonian $H_2 = A_2^\dagger A_2$ is solvable as $A_1 A_1^\dagger$ (and $A_1 A_1^\dagger - E_1^{(1)}$) is. Finally, the associated diffusion process with volatility $\sigma^{(2)}$ and drift $\mu^{(2)}$

10   PIERRE HENRY-LABORDÈRE

satisfying (20) is solvable. The superpotential $W^{(2)}(s)$ is determined by the first eigenvector of H_2, $\phi_0^{(2)}(s)$,

$$W^{(2)}(s) = -\frac{d\ln\phi_0^{(2)}(s)}{ds}$$

We can then apply a supersymmetric transformation on H_2. The new Hamiltonian H_3 can be refactorised exactly in the same way we did for H_2. Finally, it is not difficult to see that if H_1 admits p discrete (normalisable) eigenvectors, then one can generate a family of solvable Hamiltonians H_m (with a zero-eigenvalue by construction)

$$H_m = A_{m-1}A_{m-1}^\dagger - E_{m-1}^{(1)} = A_m^\dagger A_m = -\partial_s^2 + Q_m(s)$$

where $A_m = \partial_s + W_m(s)$. This corresponds to the solvable diffusion process with a drift and a volatility such that

$$W_m(s) = -\frac{d\ln\phi_0^{(m)}(s)}{ds} = \frac{1}{2}\frac{d\ln\sigma^{(m)}(s)}{ds} - \frac{\mu^{(m)}(s)}{\sqrt{2}\sigma^{(m)}(s)}$$

The eigenvalues/eigenvectors of H_m are related to those of H_{m-1} by

$$E_n^{(m)} = E_{n+1}^{(m-1)} - E_1^{(m-1)}$$
$$\phi_n^{(m)} = (E_{n+1}^{(m-1)} - E_1^{(m-1)})^{-\frac{1}{2}} A_{m-1}\phi_{n+1}^{(m-1)}$$

In particular, the superpotential of H_m is determined by the $(m-1)^{\text{th}}$ eigenvector of H_1, $\phi_{m-1}^{(1)}(s)$,

$$(21) \qquad W_m(s) \quad = \quad -\frac{d\ln(A_{m-1}\cdots A_1\phi_{m-1}^{(1)}(s))}{ds}$$

$$(22) \qquad\qquad\qquad = \quad \frac{1}{2}\frac{d\ln\sigma^{(m)}(s)}{ds} - \frac{\mu^{(m)}(s)}{\sqrt{2}\sigma^{(m)}(s)}$$

Consequently, if we know all the m discrete eigenvalues and eigenvectors of H_1, we immediately know all the energy eigenvalues and eigenfunctions of the hierarchy of $m-1$ Hamiltonians. In the following we apply this procedure starting from a known solvable superpotential, *the Natanzon superpotential*.

3.5. **Natanzon (super)potentials.** The Natanzon potential [15] is the most general potential which allows us to reduce the Schrödinger equation (8) to a Gauss or confluent hypergeometric equation (GHE or CHE).

3.5.1. *Gauss hypergeometric potential.* The potential is given by

$$Q(s) = \frac{S(z)-1}{R(z)} - \left(\frac{r_1 - 2(r_2+r_1)z}{z(1-z)} - \frac{5}{4}\frac{(r_1^2 - 4r_1r_0)}{R(z)} + r_2\right)\frac{z^2(1-z^2)}{R(z)^2}$$

with $R(z) = r_2z^2 + r_1z + r_0 > 0$ and $S(z) = s_2z^2 + s_1z + s_0$ (two second order polynomials). The z coordinate, lying in the interval $[0,1]$, is defined implicitly in terms of s by the differential equation

$$\frac{dz(s)}{ds} = \frac{2z(1-z)}{\sqrt{R(z)}}$$

Example 3.5.1. The hypergeometric Natanzon potential includes as special cases the Pösch-Teller potential II [3] $Q(s) = A + B sech(\frac{2s}{\sqrt{r_1}})^2 + C csch(\frac{2s}{\sqrt{r_1}})^2$ for $r_0 = r_2 = 0$ and the Rosen-Morse potential $Q(s) = A + B \tanh(\frac{2s}{\sqrt{r_0}}) + C sech(\frac{2s}{\sqrt{r_0}})^2$, $r_1 = r_2 = 0$ with A and B constant.

We have the discrete eigenvalues $(\alpha_n = -n \; ; n \in \mathbb{N}^*)$

$$2n + 1 = -\sqrt{1 - r_0 E_n - s_0} + \sqrt{1 - r_2 E_n - s_2} - \sqrt{1 - (r_0 + r_1 + r_2)E_n - (s_0 + s_1 + s_2)}$$

$$\psi_n(s) = (z')^{-\frac{1}{2}} z^{\frac{\gamma_n}{2}} (z - 1)^{\frac{-\gamma_n + \alpha_n + \beta_n + 1}{2}} F(\alpha_n, \beta_n, \gamma_n, z)$$

$$\gamma_n = 1 + \sqrt{1 - r_0 E_n - s_0}$$

$$\alpha_n - \beta_n = -\sqrt{1 - r_2 E_n - s_2}$$

$$\alpha_n + \beta_n = \gamma_n + \sqrt{1 - (r_0 + r_1 + r_2)E_n - (s_0 + s_1 + s_2)}$$

and the (normalized) eigenvectors

$$\psi_n(s) = B_n (z')^{-\frac{1}{2}} z^{\frac{\gamma_n}{2}} (1 - z)^{\frac{-\gamma_n - n + \beta_n + 1}{2}} P_n^{(\gamma_n - 1, -n + \beta_n - \gamma_n + 1)}(1 - 2z)$$

with $B_n = [((\frac{R(1)}{\alpha + \beta - \gamma}) + (\frac{r_0}{\gamma - 1}) - (\frac{r_2}{\beta - \alpha}))\frac{\Gamma(\gamma + n + 1)\Gamma(\alpha + \beta - \gamma)}{n!\Gamma(\beta - \alpha - n)}]^{-\frac{1}{2}}$ and $P_n^{(\gamma - 1, \alpha + \beta - \gamma)}(2z - 1)$ the Jacobi polynomials.

3.5.2. *Confluent hypergeometric potential.* A similar construction can be achieved for the class of scaled confluent hypergeometric functions. The potential is given by

$$Q(s) = \frac{S(z) - 1}{R(z)} - (\frac{r_1}{z} - \frac{5}{4} \frac{(r_1^2 - 4r_2 r_0)}{R(z)} - r_2) \frac{z^2}{R(z)^2}$$

with $R(z) = r_2 z^2 + r_1 z + r_0 > 0$ and $S(z) = s_2 z^2 + s_1 z + s_0$. The z coordinate, lying in the interval $[0, \infty[$, is defined implicitly in terms of s by the differential equation

$$\frac{dz(s)}{ds} = \frac{2z}{\sqrt{R(z)}}$$

Example 3.5.2. The confluent Natanzon potential reduces to the Morse potential $Q(s) = \frac{-1 + s_0 + s_1 e^{\frac{2s}{\sqrt{r_0}}} + s_2 e^{\frac{4s}{\sqrt{r_0}}}}{r_0}$ for $r_1 = r_2 = 0$, to the 3D oscillator $Q(s) = \frac{-\frac{3}{4} + s_0}{s^2} + \frac{s_1}{r_1} + \frac{s_2 s^2}{r_1^2}$ for $r_0 = r_2 = 0$ and to the Coulomb potential $Q(r) = \frac{-r_2 s_0 - 2s\sqrt{r_2}s_1 - 4s^2 s_2}{4r_2 s^2}$ for $r_0 = r_1 = 0$.

We have the discrete eigenvalues $\alpha_n = -n \; ; n \in \mathbb{N}$

$$\gamma_n = 1 + \sqrt{1 - r_0 E_n - s_0}$$

$$\omega_n = \sqrt{-r_2 E_n - s_2}$$

$$2n + 1 = \frac{r_1 E_n + s_1}{2\sqrt{-r_2 E_n - s_2}} - \sqrt{1 - r_0 E_n - s_0}$$

and the (normalized) eigenvectors

$$\psi_n(s) = \frac{n!}{(\gamma_n)_n} z(s)^{\frac{\gamma_n}{2}} e^{-\frac{\omega_n z(s)}{2}} (z'(s))^{-\frac{1}{2}} L_n^{\gamma_n - 1}(\omega_n z(s))$$

with $L_n^{\gamma_n - 1}(z)$ the (generalized) Laguerre polynomial. In the following, we have listed classical solvable superpotentials and the corresponding solvable local volatility models and solvable instantaneous short-rate models (Table 1. & 2.).

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[3]$sech(z) \equiv \frac{1}{\cosh(z)}$ and $csch(z) \equiv \frac{1}{\sinh(z)}$

12 PIERRE HENRY-LABORDÈRE

Superpotential	$W(s)$	$\frac{\sigma(s)}{\sigma_0}$
Shifted oscillator	$as + b$	e^{as^2+2bs}
Coulomb	$a + \frac{b}{s}$	$s^b e^{2as}$
Morse	$a + be^{-\alpha s}$	$e^{2\left(-\left(\frac{b}{\alpha e^{\alpha s}}\right)+as\right)}$
Eckart	$acoth(\alpha s) + b$	$e^{2\left(bs+\frac{a\log(\sinh(\alpha s))}{\alpha}\right)}$
Rosen-Morse	$ath(\alpha s) + b$	$e^{2bs+\frac{2a}{\alpha}\ln(\cosh(\alpha))}$
3D oscillator	$as + \frac{b}{s}$	$e^{as^2+2b\ln(s)}$
P-T I $\alpha > 2b$	$atan(\alpha s) + bcotg(\alpha s)$	$e^{2\left(-\left(\frac{a\log(\cos(\alpha s))}{\alpha}\right)+\frac{b\log(\sin(\alpha s))}{\alpha}\right)}$
P-T II $\alpha > 2b$	$ath(\alpha s) + bcoth(\alpha s)$	$e^{2\left(\frac{a\log(\cosh(\alpha s))}{\alpha}+\frac{b\log(\sinh(\alpha s))}{\alpha}\right)}$

TABLE 1. Example of solvable superpotentials-Local Volatility

one-factor short-rate model	SDE	Superpotential
Vasicek-Hull-White	$dr = k(\theta - r)dt + \sigma dW$	Shifted Osc. $(a = \frac{\kappa}{2}, b = -\frac{\kappa\theta}{\sqrt{2}\sigma})$
CIR	$dr = k(\theta - r)dt + \sigma\sqrt{r}dW$	3D Osc. $(a = \frac{\kappa}{4}, b = \frac{1}{2} - \frac{2\theta\kappa}{\sigma^2})$
Doleans	$dr = krdt + \sigma rdW$	(Constant $(W(s) = \frac{-2\kappa+\sigma^2}{2\sqrt{2}\sigma})$
EV-BK	$dr = r(\eta - \alpha\ln(r))dt + \sigma rdW$	Shifted Osc. $(a = \frac{\alpha}{2}, b = \frac{\sqrt{2}(-2\eta+\sigma^2)}{4\sigma})$

TABLE 2. Example of solvable one-factor short-rate models

3.5.3. *Natanzon hierarchy and new solvable processes.* We know that the Natanzon superpotential is related to the zero-eigenvector

$$W_{nat} = -\partial_s \ln(\psi_0(s))$$

and the corresponding supercharge A is

$$A = \partial_s + W_{nat}(s)$$

$$= \frac{2z(1-z)}{\sqrt{R}}\left(\partial_z - \frac{\gamma_0}{2z} - \frac{(1+\alpha_0+\beta_0-\gamma_0)}{2(z-1)} - \frac{\alpha_0\beta_0 F(1+\alpha_0,1+\beta_0,1+\gamma_0,z)}{\gamma_0 F(\alpha_0,\beta_0,\gamma_0,z)} + \frac{z'^{-\frac{3}{2}}z''(s)}{2}\right)$$

With a zero drift, the Natanzon superpotential corresponds to the diffusion process (15)

$$\sigma^{(1)}(s) = \frac{\sigma_0^{(1)}}{\phi_0^{(1)}(s)^2}$$

with $\sigma_0^{(1)}$ a constant of integration and $\phi_0^{(1)}(s) \equiv \psi_0(s)$. Applying the results of section (3.2), we obtain that the driftless diffusion processes

$$\sigma^{(m)}(s) = \frac{\sigma_0^{(m)}}{A_{m-1}\cdots A_1\phi_{m-1}^{(1)}(s)^2}$$

are solvable (22). Using the fact that the (discrete) eigenvector $\phi_{m-1}^{(1)}(s)$ is a hypergeometric function and that the derivative of a hypergeometric function is a new hypergeometric function [4], the action of $A_{m-1}\cdots A_1$ on $\phi_{m1}^{(1)}(s)$ will result in a sum of $(m-1)$ hypergeometric functions, thus generalizing the solution found in [1].

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[4]$_2F_1'(\alpha,\beta,\gamma,z) = \frac{\alpha\beta}{\gamma}{}_2F_1(\alpha+1,\beta+1,\gamma+1,z)$ and $M'(\alpha,\beta,z) = \frac{\alpha}{\beta}M(\alpha+1,\beta+1,z)$

4. Gauge-free stochastic volatility models

In this section, we try to identify the class of time-homogeneous stochastic volatility models which leads to an exact solution to the KBS equation. We assume that the forward f and the volatility a are driven by two correlated Brownian motions in the risk-neutral measure

$$df_t = a_t C(f_t) dW_1$$
$$da_t = b(a_t, f_t) dt + \sigma(a_t, f_t) dW_2$$
$$dW_1 dW_2 = \rho dt$$

with initial condition $a_0 = a$ and $f_0 = f$. Using the definition for the Abelian connection (5), we obtain the components

$$\mathcal{A}_f = -\frac{1}{2(1-\rho^2)} \partial_f \ln(\frac{C}{\sigma}) - \frac{\rho}{(1-\rho^2)} (\frac{b}{aC\sigma} - \frac{1}{2C} \partial_a \frac{\sigma}{a})$$

$$\mathcal{A}_a = \frac{1}{(1-\rho^2)} (\frac{b}{\sigma^2} - \frac{1}{2} \partial_a \ln(\frac{\sigma}{a}))$$

Then, the field strength is

$$\mathcal{F}_{af} = \partial_a \mathcal{A}_f - \partial_f \mathcal{A}_a$$
$$= \frac{1}{(1-\rho^2)} [(\partial_{af} \ln(\sigma) - \partial_f \frac{b}{\sigma^2}) - \rho(\frac{1}{C} \partial_a \frac{b}{a\sigma} - \frac{1}{2C} \partial_a^2 \frac{\sigma}{a} + \frac{a}{2} \partial_f^2 \frac{C}{\sigma})]$$

We will now assume that the connection is flat, $\mathcal{F}_{af} = 0$, meaning that the connection can be eliminated modulo a gauge transformation. In this case, the stochastic volatility model can thus be called a *gauge-free model*. This condition is satisfied for every correlation ρ if and only if

$$\partial_{af} \ln(\sigma) = \partial_f \frac{b}{\sigma^2}$$

$$\partial_a \frac{b}{a\sigma} - \frac{1}{2} \partial_a^2 \frac{\sigma}{a} + \frac{aC}{2} \partial_f^2 \frac{C}{\sigma} = 0$$

Moreover, if we assume that $\sigma(a)$ is only a function of a (this hypothesis is equivalent to assuming that the metric admits a Killing vector), the model is gauge-free if and only if

$$\frac{b}{\sigma} = \frac{a}{2} \partial_a \frac{\sigma}{a} + a\phi(f) - \frac{aC(f)}{2} \partial_f^2 C(f) \int^a \frac{a'da'}{\sigma(a')}$$

with $\phi(f)$ satisfying

$$\partial_f \phi(f) = \frac{\partial_f(C\partial_f^2 C)}{2} \int^a \frac{a'da'}{\sigma(a')}$$

This last equation is equivalent to $C(f)\partial_f^2 C(f) = \beta$ with β a constant and $\phi = \gamma$ a constant function. For $\beta = 0$, the above equations simplify and we have

$$C(f) = \mu f + \nu$$

$$b(a) = a\sigma(a)(\gamma + \frac{1}{2} \partial_a \frac{\sigma(a)}{a})$$

with μ, ν, γ some constant parameters. The gauge-free condition has therefore imposed the functional form of the drift term. When the volatility function is fixed respectively to a constant (Heston model), a linear function (geometric Brownian model) and a quadratic function (3/2-model) in the volatility, one obtains the correct (mean-reversion) drift [5] (see Table 7)

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[5]In order to obtain the correct number of parameters, one needs to apply a change of local time $dt = \delta dt'$, $dW_{1,2} = \sqrt{\delta} dW'_{1,2}$

14   PIERRE HENRY-LABORDÈRE

name	$\sigma(a)$	SDE
Heston	$\sigma(a) = \eta$	$dv = \delta(2v\gamma + \eta(\eta - 1))dt + 2\eta\sqrt{\delta}\sqrt{v}dW_2$
Geometric Brownian	$\sigma(a) = \eta a$	$dv = \delta(2\eta\gamma v^{\frac{3}{2}} + \eta^2 v)dt + 2\sqrt{\delta}\eta v dW_2$
3/2-model	$\sigma(a) = \eta a^2$	$dv = 2\delta\eta(\eta + \gamma)v^2 dt + 2\sqrt{\delta}\eta v^{\frac{3}{2}} dW_2$

TABLE 3. Examples of Gauge free stochastic volatility models with $df = \sqrt{\delta}(\mu + \nu f)\sqrt{v}dW_1'$.

The gauge transformation eliminating the connection is then

$$\Lambda(f,a) = -\frac{\rho^{ff}}{2}\ln(C(f)) + \alpha\rho^{fa}\int^f \frac{df'}{C(f')} + (\rho^{ff}\alpha - \frac{\rho^{af}}{2}\partial_f C)\int^a \frac{a' da'}{\sigma(a')}$$

Finally, plugging the expression for $C(f)$ and $b(a)$ into (6), we find that the function Q is [6]

$$Q = Aa^2 + B\sigma^2\partial_a\frac{a}{\sigma}$$

The Black-Scholes equation for a Vanilla option $\mathcal{C}(a, f, \tau = T - t)$ (with strike K and maturity T) satisfied by the gauge transformed function $\mathcal{C}'(a, f, \tau) = e^{\Lambda(f,a)}\mathcal{C}(a, f, \tau)$ is

$$\partial_\tau\mathcal{C}'(a, f, \tau) = \Delta\mathcal{C}'(a, f, \tau) + Q(a)\mathcal{C}'(a, f, \tau)$$

with the initial condition $\mathcal{C}'(a, f, \tau = 0) = e^{\Lambda(f,a)}(f - K)^+$. In the coordinates $q(f) = \int^f \frac{df'}{C(f')}$ and a, the Laplace-Beltrami operator is given by

$$\Delta = \frac{a\sigma}{2}(\frac{a}{\sigma}\partial_q^2 + 2\rho\partial_{aq} + \partial_a\frac{\sigma}{a}\partial_a)$$

Applying a Fourier transformation according to q, $\mathcal{C}'(q, a, \tau) = \mathcal{F}\mathcal{C}'(f, a, \tau)$, we obtain

$$\partial_\tau\mathcal{C}'(a, q, \tau) = \frac{a\sigma}{2}(-k^2\frac{a}{\sigma} + 2ik\rho\partial_a + \partial_a\frac{\sigma}{a}\partial_a)\mathcal{C}'(a, q, \tau) + Q(a)\mathcal{C}'(a, q, \tau)$$

with the initial condition $\mathcal{C}'(a, q, \tau = 0) = \mathcal{F}[e^{\Lambda(f,a)}(f - K)^+]$. Using a spectral decomposition $\mathcal{C}'(a, q, \tau) = \sum_n c_n\phi_{nk}(a)e^{-E_{nk}\tau}$ (with the coefficients c_n determined from the initial condition), the eigenvectors $\phi_{nk}(a)$ satisfy the equation

$$-E_{nk}\phi_{nk}(a) = \frac{a\sigma}{2}(-k^2\frac{a}{\sigma} + 2ik\rho\partial_a + \partial_a\frac{\sigma}{a}\partial_a)\phi_{nk}(a) + Q(a)\phi_{nk}(a)$$

This equation (23) can be further simplified by applying a *Liouville transformation* consisting in a gauge transformation and a change of variable [14]

$$\psi_{nk}(s) = (\frac{\sigma}{a})^{\frac{1}{2}}e^{ik\rho\int^a \frac{a'}{\sigma(a')}da'}\phi_{nk}(a)$$
$$\frac{ds}{da} = \frac{\sqrt{2}}{\sigma(a)}$$

$\psi_{nk}(s)$ satisfies a Schrödinger equation

$$\psi_{nk}''(s) + (E_{nk} - J(s))\psi_{nk}(s) = 0$$

---

[6]A and B are two constants given by

$A = \frac{1}{2}(-\frac{1}{2}\rho^{ff}\partial_f C + \alpha\rho^{af})^2 + \frac{1}{2}\rho^{ff}C^2\partial_f^2\ln(C) + \rho(-\frac{1}{2}\rho^{ff}\partial_f C + \alpha\rho^{af})(\alpha\rho^{ff} - \frac{\rho^{af}}{2}\partial_f C) + \frac{1}{2}(\rho^{ff}\alpha - \frac{\rho^{af}}{2}\partial_f C)^2$

$B = -\frac{1}{2}(\rho^{ff}\alpha - \rho^{af}\frac{\partial_f C}{2})$

SUPERSYMMETRIC METHODS IN OPTION PRICING   15

with the scalar potential

$$J(s) = Q(a) - \frac{k^2 a^2}{2} + \frac{4\,a^4\,k^2\,\rho^2 - 3\,\sigma(a)^2 + a^2\,\sigma'(a)^2 + 2\,a\,\sigma(a)\,(\sigma'(a) - a\,\sigma''(a))}{8\,a^2} + \frac{1}{2}\{a,s\}$$

and where the curly bracket denotes the Schwarzian derivative of a with respect to s

$$\{a,s\} = (\frac{a''(s)}{a'(s)})' - \frac{1}{2}(\frac{a''(s)}{a'(s)})^2$$

The $2d$ partial differential equation corresponding to our original KBS equation for our stochastic volatility model has therefore been reduced via a change of coordinates and a gauge transformation to a Schrödinger equation with a scalar potential $J(s)$. The stochastic volatility model is therefore solvable in terms of hypergeometric functions if the potential $J(s)$ belongs to the Natanzon class. The solution is given (in terms of the eigenvectors ψ_{nk}) by

$$\mathcal{C}(a,f,\tau) = e^{-\Lambda(a,f)}\mathcal{F}^{-1}[\sum_n \psi_{nk}(s(a))c_n e^{-E_{nk}\tau}]$$

Let us examine classical examples of solvable stochastic volatility model and show that the corresponding potentials $J(s)$ correspond to the Natanzon class (see Table 4.).

name	potential	$J(s)$
Heston	3D osc.	$J(s) = \frac{-3 + 4\,B\,s^2\,\eta + s^4\,\eta^2\left(2\,A + k^2\left(-1+\rho^2\right)\right)}{4\,s^2}$
Geometric Brownian	Morse	$J(s) = \frac{-\eta^2 + 4\,e^{\sqrt{2}\,s\,\eta}\left(2\,A + k^2\left(-1+\rho^2\right)\right)}{8}$
3/2-model	Coulombian	$J(s) = \frac{8\,A + \eta\left(-8\,B + \eta\right) + 4\,k^2\left(-1+\rho^2\right)}{4\,s^2\,\eta^2}$

TABLE 4. Stochastic volatility models and potential $J(s)$

We present here a new example of solvable stochastic volatility model which corresponds to the Pösch-Teller I potential.

Example 4.0.3 (Pösch-Teller I). For a volatility function given by $\sigma(a) = \alpha + \eta a^2$, the potential is given by a Pösch-Teller I potential [7]

$$J(s) = \frac{\alpha\left(4\left(-2A + k^2 + 4B\eta - k^2\rho^2\right) - 3\eta^2\csc(\frac{s\sqrt{\alpha}\sqrt{\eta}}{\sqrt{2}})^2 + \left(8A + \eta\left(-8B + \eta\right) + 4k^2\left(-1 + \rho^2\right)\right)\sec(\frac{s\sqrt{\alpha}\sqrt{\eta}}{\sqrt{2}})^2\right)}{8\eta}$$

5. CONCLUSION

We have shown how to use supersymmetric methods to generate new solutions to the Kolmogorov & Black-Scholes equation (KBS) for one-dimensional diffusion processes. In particular, by applying a supersymmetric transformation on the Natanzon potential, we have generated a hierarchy of new solvable processes. Then, we have classified the stochastic volatility models which admit a flat Abelian connection (with one Killing vector). The two-dimensional KBS equation has been converted into a Schrödinger equation with a scalar potential. The models for which the scalar potential belongs to the Natanzon class are solvable in terms of hypergeometric functions. This is the case for the Heston model, the geometric Brownian model and the 3/2-model. A new solution with a volatility of the volatility $\sigma(a) = \alpha + \eta a^2$, corresponding to the Pösch-Teller I, has been presented.

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[7]$\csc(z) \equiv \frac{1}{\sin(z)}$ and $\sec(z) \equiv \frac{1}{\cos(z)}$

16   PIERRE HENRY-LABORDÈRE

REFERENCES

[1] Albanese, C., Campolieti, G., Carr, P., Lipton, A., *Black-Scholes Goes Hypergeometric*, Risk Magazine, 14, Dec 2001.
[2] Albanese, C. , Kuznetsov, A., *Transformations of Markov Processes and Classification Scheme for Solvable Drift-less Diffusions*, Working Paper (2002), http://www3.imperial.ac.uk/mathfin/people/calban/papersmathfin.
[3] Albanese, C. , Kuznetsov, A., *Unifying the Three Volatility Models*, Risk Magazine (2004), Volume 17 (3), p. 94-98.
[4] Carr, P., Lipton, A., Madan, D., *Reduction Method for Valuing Derivative Securities*, Working paper (2002), http://www.math.nyu.edu/research/carrp/papers/pdf/symmetry17.pdf
[5] F. Cooper, A. Khare, U. Sukhatme, *Supersymmetry and Quantum Mechanics*, Phys. Rept. 251, 267 (1995), hep-th/9405029.
[6] Cox, J.C, *Notes on Option Pricing I: Constant Elasticity of Variance Diffusions*, Journal of Portfolio Manage-ment, 22, 15-17, 1996.
[7] Cox, J.C, Ingersoll, J.E, Ross, S.A, *A Theory of the Term Structure of Interest Rates*, Econometrica, 53, 385-407, 1985.
[8] Heston, S.: *A closed-for solution for options with stochastic volatility with applications to bond and currency options*, review of financial studies, 6, 327-343.
[9] Hull, J., White, A., *Pricing Interest Rate Derivative Securities*, The Review of Financial Studies, 3, 573-592.
[10] Junker, G., *Supersymmetric Methods in Quantum and Stastistical Physics*, Springer-Verlag (1996).
[11] Lewis, A., *Option valuation under Stochastic volatility*, Finance Press (2000).
[12] Lipton, A., *Mathematical Methods for Foreign Exchange*, World Scientific, 1st edition, 2001.
[13] Linetsky, V., *The Spectral Decomposition of the Option Value*, International Journal of Theoretical and Applied Finance, 7 (2004), pp. 337-384.
[14] Milson, R., *On the Liouville Tranformation and Exactly-Solvable Schrödinger Equations*, Internat. Journal Math., 37 (1998), 1735-1752. solv-int/9706007.
[15] Natanzon, G.A, *Study of one-dimensional Schrödinger equation generated from the hypergeometric equation*, Vestnik Leningradskogo Universiteta, 10:22, 1971, ArXiv:physics/9907032.

    E-mail address: `pierre.henry-labordere@sgcib.com`

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